ON SOME UNIVERSAL SUMS OF GENERALIZED POLYGONAL NUMBERS

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Abstract. For \( m = 3, 4, \ldots \) those \( p_m(x) = (m - 2)x(x - 1)/2 + x \) with \( x \in \mathbb{Z} \) are called generalized \( m \)-gonal numbers. Sun [S15] studied for what values of positive integers \( a, b, c \) the sum \( ap_i + bp_j + cp_k \) is universal over \( \mathbb{Z} \) (i.e., any \( n \in \mathbb{N} = \{0, 1, 2, \ldots \} \) has the form \( ap_i(x) + bp_j(y) + cp_k(z) \) with \( x, y, z \in \mathbb{Z} \)). In this paper we prove that \( p_5 + 2p_5 + 6p_5 \) are universal over \( \mathbb{Z} \), as conjectured by Sun. Sun also conjectured that any \( n \in \mathbb{N} \) can be written as \( p_3(x) + p_5(y) + p_11(z) \) and \( 3p_3(x) + p_5(y) + p_7(z) \) with \( x, y, z \in \mathbb{N} \); in contrast we show that \( p_3 + p_5 + p_11 \) and \( 3p_3 + p_5 + p_7 \) are universal over \( \mathbb{Z} \). Our proofs are essentially elementary and hence suitable for general readers.

1. Introduction

For \( m = 3, 4, \ldots \) we set

\[
p_m(x) = (m - 2)x(x - 1)/2 + x.
\]

Those \( p_m(n) \) with \( n \in \mathbb{N} = \{0, 1, 2, \ldots \} \) are the well-known \( m \)-gonal numbers (or polygonal numbers of order \( m \)). We call those \( p_m(x) \) with \( x \in \mathbb{Z} \) generalized \( m \)-gonal numbers. Note that (generalized) 3-gonal numbers are triangular numbers and (generalized) 4-gonal numbers are squares of integers.

In 1638, Fermat asserted that each \( n \in \mathbb{N} \) can be written as the sum of \( m \) polygonal numbers of order \( m \). This was proved by Lagrange, Gauss and Cauchy in the cases \( m = 4, m = 3 \) and \( m \geq 5 \) respectively (see Moreno and Wagstaff [10, pp. 54-57] or Chapter 1 of Nathanson [11, pp. 3-34]). The generalized pentagonal numbers play a crucial role in Euler’s famous recurrence for the partition function.

For \( a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots \} \) and \( i, j, k \in \{3, 4, \ldots \} \), Sun [13] called the sum \( ap_i + bp_j + cp_k \) universal over \( \mathbb{N} \) (resp., over \( \mathbb{Z} \)) if for any \( n \in \mathbb{N} \) the equation \( n = ap_i(x) + bp_j(y) + cp_k(z) \) has solutions over \( \mathbb{N} \) (resp., over \( \mathbb{Z} \)). In 1862 Liouville (cf. [4, p. 23]) determined all those universal \( ap_3 + bp_3 + cp_3 \). The second author [12] initiated the determination of those universal sums

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ap_i + bp_j + cp_k with \{i, j, k\} = \{3, 4\}, and this project was completed via [12, 5, 9]. For almost universal sums \(a p_i + b p_j + c p_k\) with \(\{i, j, k\} \subseteq \{3, 4\}\), see [8, 1, 2].

It is known that generalized hexagonal numbers are identical with triangular numbers (cf. [6] or [13, (1.3)]).

The second author recently established the following result.

**Theorem 1.1.** (Sun [13, Theorem 1.1]) Suppose that \(a p_k + b p_k + c p_k\) is universal over \(\mathbb{Z}\), where \(k \in \{4, 5, 7, 8, 9, \ldots\}\), \(a, b, c \in \mathbb{Z}^+\) and \(a \leq b \leq c\). Then \(k = 5\), \(a = 1\) and \((b, c)\) is among the following 20 ordered pairs:

\[
(1, c) \text{ (} c \in \{1, 2, 3, 4, 5, 6, 8, 9, 10\}),
(2, 2), (2, 3), (2, 4), (2, 6), (2, 8),
(3, 3), (3, 4), (3, 6), (3, 7), (3, 8), (3, 9).
\]

Guy [6] realized that \(p_5 + p_5 + p_5\) is universal over \(\mathbb{Z}\), and Sun [13] proved that the sums

\[
p_5 + p_5 + 2p_5, p_5 + p_5 + 4p_5, p_5 + 2p_5 + 2p_5,
p_5 + 2p_5 + 4p_5, p_5 + p_5 + 5p_5, p_5 + 3p_5 + 6p_5
\]

are universal over \(\mathbb{Z}\). So the converse of Theorem 1.1 reduces to the following conjecture of Sun.

**Conjecture 1.2.** (Sun [13, Remark 1.2]) The sum \(p_5 + bp_5 + cp_5\) is universal over \(\mathbb{Z}\) if the ordered pair \((b, c)\) is among

\[
(1, 3), (1, 6), (1, 8), (1, 9), (1, 10), (2, 3),
(2, 6), (2, 8), (3, 3), (3, 4), (3, 7), (3, 8), (3, 9).
\]

Our following result confirms this conjecture for six ordered pairs \((b, c)\) for the first time.

**Theorem 1.3.** For

\[
(b, c) = (1, 3), (2, 3), (2, 6), (3, 3), (3, 4), (3, 9),
\]

the sum \(p_5 + bp_5 + cp_5\) is universal over \(\mathbb{Z}\).

**Remark.** This result appeared in the initial preprint version of this paper posted to arXiv in 2009.

Sun [13] investigated those universal sums \(a p_i + b p_j + c p_k\) over \(\mathbb{N}\). By Conjectures 1.10 and 1.13 of Sun [13], \(p_3 + p_5 + p_{11}\) and \(3p_3 + p_5 + p_7\) should be universal over \(\mathbb{N}\). Though we cannot prove this, we are able to show the following result.
Theorem 1.4. The sums $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ are universal over \( \mathbb{Z} \).

Theorems 1.3 and 1.4 will be shown in Sections 2 and 3 respectively. Our proofs are essentially elementary and hence suitable for general readers.

2. Proof of Theorem 1.3

Lemma 2.1. (Sun [13, Lemma 3.2]) Let \( w = x^2 + 3y^2 \equiv 4 \pmod{8} \) with \( x, y \in \mathbb{Z} \). Then there are odd integers \( u \) and \( v \) such that \( w = u^2 + 3v^2 \).

Lemma 2.2. Let \( w = x^2 + 3y^2 \) with \( x, y \) odd and \( 3 \nmid x \). Then there are integers \( u \) and \( v \) relatively prime to 6 such that \( w = u^2 + 3v^2 \).

Proof. It suffices to consider the case \( 3 \mid y \). Without loss of generality, we may assume that \( x \not\equiv y \pmod{4} \) (otherwise we may use \(-y\) instead of \( y \)). Thus \( (x - y)/2 \) and \( (x + 3y)/2 = (x - y)/2 + 2y \) are odd. Observe that
\[
(2.1) \quad x^2 + 3y^2 = \left( \frac{x + 3y}{2} \right)^2 + 3 \left( \frac{x - y}{2} \right)^2.
\]
As \( 3 \nmid x \) and \( 3 \mid y \), neither \( (x - y)/2 \) nor \( (x + 3y)/2 \) is divisible by 3. Therefore \( u = (x + 3y)/2 \) and \( v = (x - y)/2 \) are relatively prime to 6. Thus, \( u^2 + 3v^2 \) is universal over \( \mathbb{Z} \). This concludes the proof. \( \square \)

Lemma 2.3. (Jacobi’s identity) We have
\[
(2.2) \quad 3(x^2 + y^2 + z^2) = (x + y + z)^2 + 2\left( \frac{x + y - 2z}{2} \right)^2 + 6\left( \frac{x - y}{2} \right)^2.
\]

Proof of Theorem 1.3. Let \( b, c \in \mathbb{Z}^+ \). For \( n \in \mathbb{N} \) we have
\[
n = p_5(x) + bp_5(y) + cp_5(z) = \frac{3x^2 - x}{2} + b\frac{3y^2 - y}{2} + c\frac{3z^2 - z}{2} \iff 24n + b + c + 1 = (6x - 1)^2 + b(6y - 1)^2 + c(6z - 1)^2.
\]
If \( w \in \mathbb{Z} \) is relatively prime to 6, then \( w \) or \(-w\) is congruent to \(-1\) modulo 6. Thus, \( p_5 + bp_5 + cp_5 \) is universal over \( \mathbb{Z} \) if and only if for any \( n \in \mathbb{N} \) the equation \( 24n + b + c + 1 = x^2 + by^2 + cz^2 \) has integral solutions with \( x, y, z \) relatively prime to 6.

Below we fix a nonnegative integer \( n \).

(i) By Dickson [3, Theorem III],
\[
E(x^2 + y^2 + 3z^2) = \{9^k(9l + 6) : k, l \in \mathbb{N} \}.
\]
So $24n + 5 = u^2 + v^2 + 3w^2$ for some $u, v, w \in \mathbb{Z}$. As $3w^2 \not\equiv 5 \pmod{4}$, $u$ or $v$ is odd. Without loss of generality we assume that $2 \nmid u$. Since $v^2 + 3w^2 \equiv 5 - u^2 \equiv 4 \pmod{8}$, by Lemma 2.1 we can rewrite $v^2 + 3w^2$ as $s^2 + 3t^2$ with $s, t$ odd. Now we have $24n + 5 = u^2 + s^2 + 3t^2$ with $u, s, t$ odd. By $u^2 + s^2 \equiv 5 \equiv 2 \pmod{3}$, both $u$ and $s$ are relatively prime to $3$. Applying Lemma 2.2 we can express $s^2 + 3t^2$ as $y^2 + 3z^2$ with $y, z$ relatively prime to $6$. Thus $24n + 5 = u^2 + y^2 + 3z^2$ with $u, y, z$ relatively prime to $6$. This proves the universality of $p_5 + p_5 + 3p_5$ over $\mathbb{Z}$.

(ii) By Dickson [3, Theorem X],

$$E(x^2 + 2y^2 + 3z^2) = \{4^k(16l + 10) : k, l \in \mathbb{N}\}.$$  

So $24n + 6 = 2u^2 + v^2 + 3w^2$ for some $u, v, w \in \mathbb{Z}$. Clearly $v$ and $w$ have the same parity. Thus $4 | v^2 + 3w^2$ and hence $2u^2 \equiv 6 \pmod{4}$. So $u$ is odd and $v^2 + 3w^2 \equiv 6 - 2u^2 \equiv 4 \pmod{8}$. By Lemma 2.1 we can rewrite $v^2 + 3w^2$ as $s^2 + 3t^2$ with $s, t$ odd. Now we have $24n + 6 = 2u^2 + s^2 + 3t^2$ with $u, s, t$ odd. Note that $s^2 + 2u^2 > 0$ and $s^2 + 2u^2 \equiv 0 \pmod{3}$. By [7, p. 173] or [13, Lemma 2.1], we can rewrite $s^2 + 2u^2$ as $x^2 + 2y^2$ with $x$ and $y$ relatively prime to $3$. As $x^2 + 2y^2 = s^2 + 2u^2 \equiv 3 \pmod{8}$, both $x$ and $y$ are odd. By Lemma 2.2, $x^2 + 3t^2 = r^2 + 3z^2$ for some integers $r, z \in \mathbb{Z}$ relatively prime to $6$. Thus $24n + 6 = r^2 + 2y^2 + 3z^2$ with $r, y, z$ relatively prime to $6$. It follows that $p_5 + 2p_5 + 3p_5$ is universal over $\mathbb{Z}$.

(iii) By Dickson [3, Theorem IV],

$$E(x^2 + 3y^2 + 3z^2) = \{9^k(3l + 2) : k, l \in \mathbb{N}\}.$$  

So $24n + 7 = u^2 + 3v^2 + 3w^2$ for some $u, v, w \in \mathbb{Z}$. Since $u^2 \not\equiv 7 \pmod{4}$, without loss of generality we assume that $2 \nmid w$. As $u^2 + 3v^2 \equiv 7 - 3w^2 \equiv 4 \pmod{8}$, by Lemma 2.1 there are odd integers $s$ and $t$ such that $u^2 + 3v^2 = s^2 + 3t^2$. Thus $24n + 7 = s^2 + 3t^2 + 3w^2$ with $s, t, w$ odd. Clearly, $s$ is relatively prime to $6$. By Lemma 2.2, $s^2 + 3t^2 = x_0^2 + 3y^2$ for some integers $x_0$ and $y$ relatively prime to $6$, and $x_0^2 + 3w^2 = x^2 + 3z^2$ for some integers $x$ and $z$ relatively prime to $6$. Therefore $24n + 7 = x^2 + 3y^2 + 3z^2$ with $x, y, z$ relatively prime to $6$. This proves the universality of $p_5 + 3p_5 + 3p_5$ over $\mathbb{Z}$.

(iv) By [13, Theorem 1.7(iii)], $24n + 8 = u^2 + v^2 + 3w^2$ for some $u, v, w \in \mathbb{Z}$ with $2 \nmid w$. Clearly $u \not\equiv v \pmod{2}$. Without loss of generality, we assume that $u = 2r$ with $r \in \mathbb{Z}$. Since $(2r)^2 + v^2 \equiv 8 \equiv 2 \pmod{3}$, both $r$ and $v$ are relatively prime to $3$. As $v$ and $w$ are odd, $v^2 + 3w^2 \equiv 4 \pmod{8}$ and hence $r$ is odd. By Lemma 2.2, we can rewrite $v^2 + 3w^2$ as $x^2 + 3y^2$ with $x$ and $y$ relatively prime to $6$. Note that $24n + 8 = 4r^2 + v^2 + 3w^2 = x^2 + 3y^2 + 4r^2$.  

with \(x, y, r\) relatively prime to 6. It follows that \(p_5 + 3p_5 + 4p_5\) is universal over \(\mathbb{Z}\).

(v) By (2.3), \(24n + 13 = u^2 + v^2 + 3w^2\) for some \(u, v, w \in \mathbb{Z}\). Since \(3w^2 \not\equiv 13 \equiv 1 \pmod{4}\), without loss of generality we may assume that \(u\) is odd. As \(v^2 + 3w^2 \equiv 13 - u^2 \equiv 4 \pmod{8}\), by Lemma 2.1 we can rewrite \(v^2 + 3w^2\) as \(s^2 + 3t^2\) with \(s\) and \(t\) odd. Thus \(24n + 13 = u^2 + s^2 + 3t^2\) with \(u, s, t\) odd. Since \(u^2 + s^2 \equiv 13 \equiv 1 \pmod{3}\), without loss of generality we may assume that \(3 \nmid u\) and \(s = 3r\) with \(r \in \mathbb{Z}\). By Lemma 2.2, \(u^2 + 3r^2 = x^2 + 3y_0^2\) for some integers \(x\) and \(y_0\) relatively prime to 6, also \(y_0^2 + 3r^2 = y^2 + 3z^2\) for some integers \(y\) and \(z\) relatively prime to 6. Thus \(24n + 13 = x^2 + 3y_0^2 + 9r^2 = x^2 + 3y^2 + 9z^2\) with \(x, y, z\) relatively prime to 6. This proves the universality of \(p_5 + 3p_5 + 9p_5\) over \(\mathbb{Z}\).

(vi) By the Gauss-Legendre theorem (cf. [11, pp. 17-23]), \(8n + 3 = x^2 + y^2 + z^2\) for some odd integers \(x, y, z\). Without loss of generality we may assume that \(x \not\equiv y \pmod{4}\). By Jacobi’s identity (2.2), we have \(3(8n + 3) = u^2 + 2v^2 + 6w^2\), where \(u = x + y + z\), \(v = (x + y)/2 - z\) and \(w = (x - y)/2\) are odd integers. As \(u^2 + 2v^2\) is a positive integer divisible by 3, by [7, p. 173] or [13, Lemma 2.1] we can write \(u^2 + 2v^2 = a^2 + 2b^2\) with \(a\) and \(b\) relatively prime to 3. Since \(a^2 + 2b^2 = u^2 + 2v^2 \equiv 3 \pmod{8}\), both \(a\) and \(b\) are odd. By Lemma 2.2, \(b^2 + 3w^2 = c^2 + 3d^2\) for some integers \(c\) and \(d\) relatively prime to 6. Thus \(24n + 9 = a^2 + 2b^2 + 6w^2 = a^2 + 2c^2 + 6d^2\) with \(a, c, d\) relatively prime to 6. It follows that \(p_5 + 2p_5 + 6p_5\) is universal over \(\mathbb{Z}\).

In view of the above, we have completed the proof of Theorem 1.3. \(\square\)

### 3. Proof of Theorem 1.4

**Proof of Theorem 1.4.** (i) Let \(n \in \mathbb{N}\). By part (v) in the proof of Theorem 1.3, there are integers \(u, v, w \in \mathbb{Z}\) relatively prime to 6 such that

\[
72n + 61 = 24(3n + 2) + 13 = 9u^2 + 3v^2 + w^2.
\]

Clearly \(w^2 \equiv 61 - 3v^2 \equiv 7^2 \pmod{9}\) and hence \(w \equiv \pm 7 \pmod{9}\). So there are \(x, y, z \in \mathbb{Z}\) such that

\[
72n + 61 = 9(2x + 1)^2 + 3(6y - 1)^2 + (18z - 7)^2
\]

and hence \(n = p_3(x) + p_5(y) + p_{11}(z)\). (Note that \(p_{11}(x) = 9(x^2 - x)/2 + x = (9x^2 - 7x)/2\).)

(ii) Let \(n \in \mathbb{N}\). It is easy to see that

\[
n = 3p_3(x) + p_5(y) + p_7(z)
\]

\(\iff\) \(120n + 77 = 5(3(2x + 1))^2 + 5(6y - 1)^2 + 3(10z - 3)^2\).
Suppose \(120n + 77 = 5x^2 + 5y^2 + 3z^2\) for some \(x, y, z \in \mathbb{Z}\) with \(z\) odd. Then \(x^2 + y^2 \equiv 77 - 3z^2 \equiv 2 \pmod{4}\) and hence \(x\) and \(y\) are odd. Note that \(3z^2 \equiv 77 \equiv 12 \pmod{5}\) and hence \(z \equiv \pm 3 \pmod{10}\). As \(5x^2 + 5y^2 \equiv 77 \equiv 5 \pmod{3}\), exactly one of \(x\) and \(y\) is divisible by 3. Thus there are \(u, v, w \in \mathbb{Z}\) such that

\[
120n + 77 = 5(3(2u + 1))^2 + 5(6v - 1)^2 + 3(10w - 3)^2.
\]

By the above, to prove the universality of \(3p_3 + p_5 + p_7\) over \(\mathbb{Z}\), we only need to show that \(120n + 77 = 5x^2 + 5y^2 + 3z^2\) for some \(x, y, z \in \mathbb{Z}\) with \(z\) odd.

By (2.3), there are \(u, v, w \in \mathbb{Z}\) such that \(120n + 77 = u^2 + v^2 + 3w^2\). As \(3w^2 \not\equiv 77 \equiv 1 \pmod{4}\), \(u\) or \(v\) is odd, say, \(2 \nmid u\). As \(v^2 + 3w^2 \equiv 77 - u^2 \equiv 4 \pmod{8}\), by Lemma 2.1 we may assume that \(v\) and \(w\) are odd without loss of generality.

We claim that \(120n + 77 = a^2 + b^2 + 3c^2\) for some odd integers \(a, b, c\) with \(c \equiv \pm 2 \pmod{5}\). This holds if \(w \equiv \pm 2 \pmod{5}\). Suppose that \(w \not\equiv \pm 2 \pmod{5}\). If \(w \equiv \pm 1 \pmod{5}\), then \(u^2 + v^2 \equiv 77 - 3w^2 \equiv -1 \pmod{5}\) and hence \(u\) or \(v\) is divisible by 5. If \(w \equiv 0 \pmod{5}\), then \(u^2 + v^2 \equiv 77 \equiv 2 \pmod{5}\) and hence \(u^2 \equiv v^2 \equiv 1 \pmod{5}\). Without loss of generality, we assume that one of \(v\) and \(w\) is divisible by 5 and the other one is congruent to 1 or \(-1\) modulo 5, we may also suppose that \(v \not\equiv w \pmod{4}\) (otherwise we may use \(-w\) instead of \(w\)). By the identity (2.1),

\[
 v^2 + 3w^2 = \left(\frac{v + 3w}{2}\right)^2 + 3\left(\frac{v - w}{2}\right)^2.
\]

Note that both \((v - w)/2\) and \((v + 3w)/2 = (v - w)/2 + 2w\) are odd. Also, \((v - w)/2\) is congruent to 2 or \(-2\) modulo 5. This confirms the claim.

By the above, there are odd integers \(a, b, c \in \mathbb{Z}\) with \(c \equiv \pm 2 \pmod{5}\) such that \(120n + 77 = a^2 + b^2 + 3c^2\). Since \(3c^2 \equiv 77 \pmod{5}\), we have \(5 \mid a^2 + b^2\) and hence \(a^2 \equiv (2b)^2 \pmod{5}\). Without loss of generality we assume that \(a \equiv 2b \pmod{5}\). Then \(x = (2a + b)/5\) and \(y = (a - 2b)/5\) are odd integers, and

\[
 a^2 + b^2 = (2x + y)^2 + (x - 2y)^2 = 5(x^2 + y^2).
\]

Now we have \(120n + 77 = 5(x^2 + y^2) + 3c^2\) with \(x, y, c\) odd. This concludes our proof.

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