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ON SOME UNIVERSAL SUMS OF GENERALIZED POLYGONAL NUMBERS

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ABSTRACT. For $m = 3, 4, \ldots$ those $p_m(x) = (m-2)x(x-1)/2 + x$ with $x \in \mathbb{Z}$ are called generalized m-gonal numbers. Sun [S15] studied for what values of positive integers a, b, c the sum $ap_5+bp_5+cp_5$ is universal over $\mathbb{Z}$ (i.e., any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ has the form $ap_5(x)+bp_5(y)+cp_5(z)$ with $x, y, z \in \mathbb{Z}$). In this paper we prove that $p_5 + bp_5 + 3p_5$ ($b = 1, 2, 3, 4, 9$) and $p_5 + 2p_5 + 6p_5$ are universal over $\mathbb{Z}$, as conjectured by Sun. Sun also conjectured that any $n \in \mathbb{N}$ can be written as $p_3(x) + p_5(y) + p_{11}(z)$ and $3p_3(x) + p_5(y) + p_7(z)$ with $x, y, z \in \mathbb{N}$; in contrast we show that $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ are universal over $\mathbb{Z}$. Our proofs are essentially elementary and hence suitable for general readers.

1. Introduction

For $m = 3, 4, \ldots$ we set

$$p_m(x) = (m-2)\frac{x(x-1)}{2} + x. \tag{1.1}$$

Those $p_m(n)$ with $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ are the well-known *m-gonal numbers* (or polygonal numbers of order m). We call those $p_m(x)$ with $x \in \mathbb{Z}$ *generalized m-gonal numbers*. Note that (generalized) 3-gonal numbers are triangular numbers and (generalized) 4-gonal numbers are squares of integers.

In 1638, Fermat asserted that each $n \in \mathbb{N}$ can be written as the sum of m polygonal numbers of order m. This was proved by Lagrange, Gauss and Cauchy in the cases $m = 4$, $m = 3$ and $m \geqslant 5$ respectively (see Moreno and Wagstaff [10, pp. 54-57] or Chapter 1 of Nathanson [11, pp. 3-34]). The generalized pentagonal numbers play a crucial role in Euler's famous recurrence for the partition function.

For $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $i, j, k \in \{3, 4, \ldots\}$, Sun [13] called the sum $ap_i + bp_j + cp_k$ universal over $\mathbb{N}$ (resp., over $\mathbb{Z}$) if for any $n \in \mathbb{N}$ the equation $n = ap_i(x)+bp_j(y)+cp_k(z)$ has solutions over $\mathbb{N}$ (resp., over $\mathbb{Z}$). In 1862 Liouville (cf. [4, p. 23]) determined all those universal $ap_3 + bp_3 + cp_3$. The second author [12] initiated the determination of those universal sums

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$ap_i + bp_j + cp_k$ with $\{i, j, k\} = \{3, 4\}$, and this project was completed via [12, 5, 9]. For almost universal sums $ap_i + bp_j + cp_k$ with $\{i, j, k\} \subseteq \{3, 4\}$, see [8, 1, 2].

It is known that generalized hexagonal numbers are identical with triangular numbers (cf. [6] or [13, (1.3)]).

The second author recently established the following result.

Theorem 1.1. (Sun [13, Theorem 1.1]) *Suppose that $ap_k + bp_k + cp_k$ is universal over $\mathbb{Z}$, where $k \in \{4, 5, 7, 8, 9, \ldots\}$, $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then $k = 5$, $a = 1$ and (b, c) is among the following 20 ordered pairs:*

$$(1, c)\ (c \in \{1, 2, 3, 4, 5, 6, 8, 9, 10\}),$$
$$(2, 2),\ (2, 3),\ (2, 4),\ (2, 6),\ (2, 8),$$
$$(3, 3),\ (3, 4),\ (3, 6),\ (3, 7),\ (3, 8),\ (3, 9).$$

Guy [6] realized that $p_5+p_5+p_5$ is universal over $\mathbb{Z}$, and Sun [13] proved that the sums

$$p_5 + p_5 + 2p_5,\ p_5 + p_5 + 4p_5,\ p_5 + 2p_5 + 2p_5,$$
$$p_5 + 2p_5 + 4p_5,\ p_5 + p_5 + 5p_5,\ p_5 + 3p_5 + 6p_5$$

are universal over $\mathbb{Z}$. So the converse of Theorem 1.1 reduces to the following conjecture of Sun.

Conjecture 1.2. (Sun [13, Remark 1.2]) *The sum $p_5+bp_5+cp_5$ is universal over $\mathbb{Z}$ if the ordered pair (b, c) is among*

$$(1, 3),\ (1, 6),\ (1, 8),\ (1, 9),\ (1, 10),\ (2, 3),$$
$$(2, 6),\ (2, 8),\ (3, 3),\ (3, 4),\ (3, 7),\ (3, 8),\ (3, 9).$$

Our following result confirms this conjecture for six ordered pairs (b, c) for the first time.

Theorem 1.3. *For*

$$(b, c) = (1, 3),\ (2, 3),\ (2, 6),\ (3, 3),\ (3, 4),\ (3, 9),$$

the sum $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$.

Remark. This result appeared in the initial preprint version of this paper posted to `arXiv` in 2009.

Sun [13] investigated those universal sums $ap_i + bp_j + cp_k$ over $\mathbb{N}$. By Conjectures 1.10 and 1.13 of Sun [13], $p_3+p_5+p_{11}$ and $3p_3+p_5+p_7$ should be universal over $\mathbb{N}$. Though we cannot prove this, we are able to show the following result.

Theorem 1.4. *The sums $p_3 + p_5 + p_{11}$ and $3p_3 + p_5 + p_7$ are universal over $\mathbb{Z}$.*

Theorems 1.3 and 1.4 will be shown in Sections 2 and 3 respectively. Our proofs are essentially elementary and hence suitable for general readers.

2. Proof of Theorem 1.3

Lemma 2.1. (Sun [13, Lemma 3.2]) *Let $w = x^2 + 3y^2 \equiv 4 \pmod 8$ with $x, y \in \mathbb{Z}$. Then there are odd integers u and v such that $w = u^2 + 3v^2$.*

Lemma 2.2. *Let $w = x^2 + 3y^2$ with x, y odd and $3 \nmid x$. Then there are integers u and v relatively prime to 6 such that $w = u^2 + 3v^2$.*

Proof. It suffices to consider the case $3 \mid y$. Without loss of generality, we may assume that $x \not\equiv y \pmod 4$ (otherwise we may use $-y$ instead of y). Thus $(x - y)/2$ and $(x + 3y)/2 = (x - y)/2 + 2y$ are odd. Observe that

$$x^2 + 3y^2 = \left(\frac{x + 3y}{2}\right)^2 + 3\left(\frac{x - y}{2}\right)^2. \tag{2.1}$$

As $3 \nmid x$ and $3 \mid y$, neither $(x-y)/2$ nor $(x+3y)/2$ is divisible by 3. Therefore $u = (x + 3y)/2$ and $v = (x - y)/2$ are relatively prime to 6. This concludes the proof. $\square$

Lemma 2.3. (Jacobi's identity) *We have*

$$3(x^2 + y^2 + z^2) = (x + y + z)^2 + 2\left(\frac{x + y - 2z}{2}\right)^2 + 6\left(\frac{x - y}{2}\right)^2. \tag{2.2}$$

We need to introduce some notation. For $a, b, c \in \mathbb{Z}^+$, we set

$$E(ax^2 + by^2 + cz^2) = \{n \in \mathbb{N} : \ n \neq ax^2 + by^2 + cz^2 \text{ for any } x, y, z \in \mathbb{Z}\}.$$

Proof of Theorem 1.3. Let $b, c \in \mathbb{Z}^+$. For $n \in \mathbb{N}$ we have

$$\begin{aligned} & n = p_5(x) + bp_5(y) + cp_5(z) = \frac{3x^2 - x}{2} + b\frac{3y^2 - y}{2} + c\frac{3z^2 - z}{2} \\ \iff & 24n + b + c + 1 = (6x - 1)^2 + b(6y - 1)^2 + c(6z - 1)^2. \end{aligned}$$

If $w \in \mathbb{Z}$ is relatively prime to 6, then w or $-w$ is congruent to -1 modulo 6. Thus, $p_5 + bp_5 + cp_5$ is universal over $\mathbb{Z}$ if and only if for any $n \in \mathbb{N}$ the equation $24n + b + c + 1 = x^2 + by^2 + cz^2$ has integral solutions with x, y, z relatively prime to 6.

Below we fix a nonnegative integer n.

(i) By Dickson [3, Theorem III],

$$E(x^2 + y^2 + 3z^2) = \{9^k(9l + 6) : \ k, l \in \mathbb{N}\}. \tag{2.3}$$

So $24n+5=u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. As $3w^2\not\equiv 5\pmod 4$, u or v is odd. Without loss of generality we assume that $2\nmid u$. Since $v^2+3w^2\equiv 5-u^2\equiv 4\pmod 8$, by Lemma 2.1 we can rewrite v^2+3w^2 as s^2+3t^2 with s,t odd. Now we have $24n+5=u^2+s^2+3t^2$ with u,s,t odd. By $u^2+s^2\equiv 5\equiv 2\pmod 3$, both u and s are relatively prime to 3. Applying Lemma 2.2 we can express s^2+3t^2 as y^2+3z^2 with y,z relatively prime to 6. Thus $24n+5=u^2+y^2+3z^2$ with u,y,z relatively prime to 6. This proves the universality of $p_5+p_5+3p_5$ over $\mathbb{Z}$.

(ii) By Dickson [3, Theorem X],

$$E(x^2+2y^2+3z^2)=\{4^k(16l+10):\ k,l\in\mathbb{N}\}. \tag{2.4}$$

So $24n+6=2u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Clearly v and w have the same parity. Thus $4\mid v^2+3w^2$ and hence $2u^2\equiv 6\pmod 4$. So u is odd and $v^2+3w^2\equiv 6-2u^2\equiv 4\pmod 8$. By Lemma 2.1 we can rewrite v^2+3w^2 as s^2+3t^2 with s,t odd. Now we have $24n+6=2u^2+s^2+3t^2$ with u,s,t odd. Note that $s^2+2u^2>0$ and $s^2+2u^2\equiv 0\pmod 3$. By [7, p. 173] or [13, Lemma 2.1], we can rewrite s^2+2u^2 as x^2+2y^2 with x and y relatively prime to 3. As $x^2+2y^2=s^2+2u^2\equiv 3\pmod 8$, both x and y are odd. By Lemma 2.2, $x^2+3t^2=r^2+3z^2$ for some integers $r,z\in\mathbb{Z}$ relatively prime to 6. Thus $24n+6=r^2+2y^2+3z^2$ with r,y,z relatively prime to 6. It follows that $p_5+2p_5+3p_5$ is universal over $\mathbb{Z}$.

(iii) By Dickson [3, Theorem IV],

$$E(x^2+3y^2+3z^2)=\{9^k(3l+2):\ k,l\in\mathbb{N}\}. \tag{2.5}$$

So $24n+7=u^2+3v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$. Since $u^2\not\equiv 7\pmod 4$, without loss of generality we assume that $2\nmid w$. As $u^2+3v^2\equiv 7-3w^2\equiv 4\pmod 8$, by Lemma 2.1 there are odd integers s and t such that $u^2+3v^2=s^2+3t^2$. Thus $24n+7=s^2+3t^2+3w^2$ with s,t,w odd. Clearly, s is relatively prime to 6. By Lemma 2.2, $s^2+3t^2=x_0^2+3y^2$ for some integers x_0 and y relatively prime to 6, and $x_0^2+3w^2=x^2+3z^2$ for some integers x and z relatively prime to 6. Therefore $24n+7=x^2+3y^2+3z^2$ with x,y,z relatively prime to 6. This proves the universality of $p_5+3p_5+3p_5$ over $\mathbb{Z}$.

(iv) By [13, Theorem 1.7(iii)], $24n+8=u^2+v^2+3w^2$ for some $u,v,w\in\mathbb{Z}$ with $2\nmid w$. Clearly $u\not\equiv v\pmod 2$. Without loss of generality, we assume that $u=2r$ with $r\in\mathbb{Z}$. Since $(2r)^2+v^2\equiv 8\equiv 2\pmod 3$, both r and v are relatively prime to 3. As v and w are odd, $v^2+3w^2\equiv 4\pmod 8$ and hence r is odd. By Lemma 2.2, we can rewrite v^2+3w^2 as x^2+3y^2 with x and y relatively prime to 6. Note that $24n+8=4r^2+v^2+3w^2=x^2+3y^2+4r^2$

with x, y, r relatively prime to 6. It follows that $p_5 + 3p_5 + 4p_5$ is universal over $\mathbb{Z}$.

(v) By (2.3), $24n + 13 = u^2 + v^2 + 3w^2$ for some $u, v, w \in \mathbb{Z}$. Since $3w^2 \not\equiv 13 \equiv 1 \pmod{4}$, without loss of generality we may assume that u is odd. As $v^2 + 3w^2 \equiv 13 - u^2 \equiv 4 \pmod{8}$, by Lemma 2.1 we can rewrite v^2+3w^2 as s^2+3t^2 with s and t odd. Thus $24n+13 = u^2+s^2+3t^2$ with u, s, t odd. Since $u^2 + s^2 \equiv 13 \equiv 1 \pmod{3}$, without loss of generality we may assume that $3 \nmid u$ and $s = 3r$ with $r \in \mathbb{Z}$. By Lemma 2.2, $u^2+3t^2 = x^2+3y_0^2$ for some integers x and y_0 relatively prime to 6, also $y_0^2 + 3r^2 = y^2 + 3z^2$ for some integers y and z relatively prime to 6. Thus $24n+13 = x^2+3y_0^2+9r^2 = x^2+3y^2+9z^2$ with x, y, z relatively prime to 6. This proves the universality of $p_5 + 3p_5 + 9p_5$ over $\mathbb{Z}$.

(vi) By the Gauss-Legendre theorem (cf. [11, pp. 17-23]), $8n + 3 = x^2 + y^2 + z^2$ for some odd integers x, y, z. Without loss of generality we may assume that $x \not\equiv y \pmod{4}$. By Jacobi's identity (2.2), we have $3(8n+3) = u^2+2v^2+6w^2$, where $u = x+y+z$, $v = (x+y)/2-z$ and $w = (x-y)/2$ are odd integers. As $u^2 + 2v^2$ is a positive integer divisible by 3, by [7, p. 173] or [13, Lemma 2.1] we can write $u^2 + 2v^2 = a^2 + 2b^2$ with a and b relatively prime to 3. Since $a^2+2b^2 = u^2+2v^2 \equiv 3 \pmod{8}$, both a and b are odd. By Lemma 2.2, $b^2 + 3w^2 = c^2 + 3d^2$ for some integers c and d relatively prime to 6. Thus $24n + 9 = a^2 + 2b^2 + 6w^2 = a^2 + 2c^2 + 6d^2$ with a, c, d relatively prime to 6. It follows that $p_5 + 2p_5 + 6p_5$ is universal over $\mathbb{Z}$.

In view of the above, we have completed the proof of Theorem 1.3. $\square$

3. Proof of Theorem 1.4

Proof of Theorem 1.4. (i) Let $n \in \mathbb{N}$. By part (v) in the proof of Theorem 1.3, there are integers $u, v, w \in \mathbb{Z}$ relatively prime to 6 such that

$$72n + 61 = 24(3n + 2) + 13 = 9u^2 + 3v^2 + w^2.$$

Clearly $w^2 \equiv 61 - 3v^2 \equiv 7^2 \pmod{9}$ and hence $w \equiv \pm 7 \pmod{9}$. So there are $x, y, z \in \mathbb{Z}$ such that

$$72n + 61 = 9(2x + 1)^2 + 3(6y - 1)^2 + (18z - 7)^2$$

and hence $n = p_3(x)+p_5(y)+p_{11}(z)$. (Note that $p_{11}(x) = 9(x^2-x)/2+x = (9x^2 - 7x)/2$.)

(ii) Let $n \in \mathbb{N}$. It is easy to see that

$$\begin{aligned} & n = 3p_3(x) + p_5(y) + p_7(z) \\ \iff & 120n + 77 = 5(3(2x + 1))^2 + 5(6y - 1)^2 + 3(10z - 3)^2. \end{aligned}$$

Suppose $120n+77=5x^2+5y^2+3z^2$ for some $x,y,z\in\mathbb{Z}$ with z odd. Then $x^2+y^2\equiv 77-3z^2\equiv 2\pmod 4$ and hence x and y are odd. Note that $3z^2\equiv 77\equiv 12\pmod 5$ and hence $z\equiv\pm3\pmod{10}$. As $5x^2+5y^2\equiv 77\equiv 5\pmod 3$, exactly one of x and y is divisible by 3. Thus there are $u,v,w\in\mathbb{Z}$ such that

$$120n+77=5(3(2u+1))^2+5(6v-1)^2+3(10w-3)^2.$$

By the above, to prove the universality of $3p_3+p_5+p_7$ over $\mathbb{Z}$, we only need to show that $120n+77=5x^2+5y^2+3z^2$ for some $x,y,z\in\mathbb{Z}$ with z odd.

By (2.3), there are $u,v,w\in\mathbb{Z}$ such that $120n+77=u^2+v^2+3w^2$. As $3w^2\not\equiv 77\equiv 1\pmod 4$, u or v is odd, say, $2\nmid u$. As $v^2+3w^2\equiv 77-u^2\equiv 4\pmod 8$, by Lemma 2.1 we may assume that v and w are odd without loss of generality.

We claim that $120n+77=a^2+b^2+3c^2$ for some odd integers a,b,c with $c\equiv\pm2\pmod 5$. This holds if $w\equiv\pm2\pmod 5$. Suppose that $w\not\equiv\pm2\pmod 5$. If $w\equiv\pm1\pmod 5$, then $u^2+v^2\equiv 77-3w^2\equiv -1\pmod 5$ and hence u or v is divisible by 5. If $w\equiv 0\pmod 5$, then $u^2+v^2\equiv 77\equiv 2\pmod 5$ and hence $u^2\equiv v^2\equiv 1\pmod 5$. Without loss of generality, we assume that one of v and w is divisible by 5 and the other one is congruent to 1 or -1 modulo 5, we may also suppose that $v\not\equiv w\pmod 4$ (otherwise we may use $-w$ instead of w). By the identity (2.1),

$$v^2+3w^2=\left(\frac{v+3w}{2}\right)^2+3\left(\frac{v-w}{2}\right)^2.$$

Note that both $(v-w)/2$ and $(v+3w)/2=(v-w)/2+2w$ are odd. Also, $(v-w)/2$ is congruent to 2 or -2 modulo 5. This confirms the claim.

By the above, there are odd integers $a,b,c\in\mathbb{Z}$ with $c\equiv\pm2\pmod 5$ such that $120n+77=a^2+b^2+3c^2$. Since $3c^2\equiv 77\pmod 5$, we have $5\mid a^2+b^2$ and hence $a^2\equiv(2b)^2\pmod 5$. Without loss of generality we assume that $a\equiv 2b\pmod 5$. Then $x=(2a+b)/5$ and $y=(a-2b)/5$ are odd integers, and

$$a^2+b^2=(2x+y)^2+(x-2y)^2=5(x^2+y^2).$$

Now we have $120n+77=5(x^2+y)^2+3c^2$ with x,y,c odd. This concludes our proof. $\square$

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References

[1] W. K. Chan and A. Haensch, *Alomost universal ternary sums of squares and triangular numbers*, in: Quadratic and Higher Degree Forms, Dev. Math., Vol. 31, Springer, New York, 2013, pp. 51–62.

[2] W. K. Chan and B.-K. Oh, *Almost universal ternary sums of triangular numbers*, Proc. Amer. Math. Soc. **137** (2009), 3553–3562.

[3] L. E. Dickson, *Integers represented by positive ternary quadratic forms*, Bull. Amer. Math. Soc. **33** (1927), 63–77.

[4] L. E. Dickson, History of the Theory of Numbers, Vol. II, AMS Chelsea Publ., 1999.

[5] S. Guo, H. Pan and Z.-W. Sun, *Mixed sums of squares and triangular numbers (II)*, Integers **7** (2007), #A56, 5pp (electronic).

[6] R. K. Guy, *Every number is expressible as the sum of how many polygonal numbers?* Amer. Math. Monthly **101** (1994), 169–172.

[7] B. W. Jones and G. Pall, *Regular and semi-regular positive ternary quadratic forms*, Acta Math. **70** (1939), 165–191.

[8] B. Kane and Z.-W. Sun, *On almost universal mixed sums of squares and triangular numbers*, Trans. Amer. Math. Soc. **362** (2010), 6425–6455.

[9] B.-K. Oh and Z.-W. Sun, *Mixed sums of squares and triangular numbers (III)*, J. Number Theory **129** (2009), 964-969.

[10] C. J. Moreno and S. S. Wagstaff, Sums of Squares of Integers, Chapman & Hall/CRC, New York, 2005.

[11] M. B. Nathanson, Additive Number Theory: The Classical Bases, Grad. Texts in Math., vol. 164, Springer, New York, 1996.

[12] Z.-W. Sun, *Mixed sums of squares and triangular numbers*, Acta Arith. **127** (2007), 103–113.

[13] Z.-W. Sun, *On universal sums of polygonal numbers*, Sci. China Math. **58** (2015), 1367–1396.

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