THE NUMBER OF ZEROS OF $L'(s,\chi)$

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Abstract. Assuming the generalized Riemann Hypothesis, we show that for $q > 1$ and $T > 2$

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi e} + O\left(\frac{\log qT}{\log\log qT} + \sqrt{m\log 2m\log qT}\right).$$

where $N_1(T,\chi)$ is the number of zeros of $L'(s,\chi)$ in the region $\Re s > 0$, $|\Im s| \le T$, χ is a primitive character to the modulus q, m is the smallest prime number not dividing q, and the implied constant is absolute.

1. Introduction

A basic question in the classical theory of L-functions is the zero counting problem. For example, for the Riemann zeta-function $\zeta(s)$, the Riemann-von Mangoldt formula (see Theorem 9.4 in [12]) states that for $T \ge 2$

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T), \tag{1.1}$$

where $N(T)$ is the number of zeros of $\zeta(s)$ in the region $\Re s > 0$, $0 < \Im s \le T$. As another example, let $q > 1$ be an integer and χ be a primitive character to the modulus q. Let $N(T,\chi)$ denote the number of zeros of the Dirichlet L-function $L(s,\chi)$ in the region $\Re s > 0$, $|\Im s| \le T$. Then we know (see Chapter 16 in [4]) that for $T \ge 2$

$$N(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi e} + O(\log qT). \tag{1.2}$$

Here and throughout, the implied constants are always independent of q, χ and T.

There has also been considerable interest in the zeros of the first derivative of the Riemann zeta-function, because they are closely connected to the distribution of zeros of zeta itself. As just one illustration we cite Speiser's [10] theorem that the Riemann Hypothesis is equivalent to the nonexistence of non-real zeros of $\zeta'(s)$ in the half-plane $\Re s < 1/2$. Recently, Akatsuka and Suriajaya [2] have obtained analogues of Speiser's theorem in the case of Dirichlet L-functions.

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Let $N_1(T)$ denote the number of zeros of $\zeta'(s)$ in the region $\Re s > 0$, $0 < \Im s \le T$. In [3] Berndt proved that

$$N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O(\log T) \tag{1.3}$$

as $T \to \infty$. This should be compared with the Riemann-von Mangoldt formula (1.1). In the Dirichlet L-functions setting, Akatsuka and Suriajaya [2] showed that

$$N_1(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2m\pi e} + O(m^{1/2} \log qT), \tag{1.4}$$

where $N_1(T, \chi)$ is the number of zeros of $L'(s, \chi)$ in the region $\Re s > 0$, $|\Im s| \le T$, and $m = m(q)$ is the smallest prime number not dividing q. The formula (1.4) improves a previous result of Yıldırım [13]. One should notice that the error term in (1.4) is of a factor $m^{1/2}$ larger than that in (1.2). It is clear that for odd q we have $m = 2$; also, a simple calculation shows that for any large prime p, if q is the product of all the prime numbers less than p, then $m = p$ is of size $\log q$. On the contrary, it is easy to prove that $m = O(\log q)$.

It is natural to wonder what are the true sizes of the error terms in above formulas (1) – (4). Using interesting heuristic arguments, Farmer, Gonek and Hughes [5] have conjectured that the error term in (1.1) is $O(\sqrt{\log T \log\log T})$. Assuming the Riemann Hypothesis (RH), Littlewood [7] showed that the error term in (1.1) may be replaced by

$$O\left(\frac{\log T}{\log\log T}\right). \tag{1.5}$$

For the derivative of zeta, the author [6] has recently proved that on RH the error term in (1.3) may also be reduced to (1.5), which improves previous work of Akatsuka [1]. In the Dirichlet L-function case, it follows from Selberg's work [9] that on the generalized Riemann Hypothesis (GRH) the error term in (1.2) can be bounded by

$$O\left(\frac{\log qT}{\log\log qT}\right). \tag{1.6}$$

Regarding the error term in (1.4), Suriajaya [11] proved that on GRH it can be replaced by

$$O\left(\log q + \frac{m^{1/2} \log qT}{\log\log qT} \cdot \min\left\{\sqrt{\log\log qT}, 1 + \frac{m^{1/2}}{\log\log qT}\right\}\right). \tag{1.7}$$

Our purpose here is to show that if GRH is true, the error term in (1.4) can be reduced to

$$O\left(\frac{\log qT}{\log\log qT}+\sqrt{m\log 2m\log qT}\right).$$

Thus it improves (1.7).

Theorem 1.1. *Assume GRH. For $q>1$ and $T>2$ we have*

$$N_1(T,\chi)=\frac{T}{\pi}\log\frac{qT}{2m\pi e}+O\left(\frac{\log qT}{\log\log qT}+\sqrt{m\log 2m\log qT}\right).$$

Recall that $m=O(\log q)$. Thus, our theorem shows in particular that on GRH, the power of $\log q$ in the error term for $N_1(T,\chi)$ is no greater than 1, uniformly for all $q>1$. On the other hand, since m actually reaches the size $\log q$ infinitely often, the power of $\log q$ can be as large as $3/2$ in the upper bound of the error term in the unconditional formula (1.4). In view of these observations, it would be of interest to see whether we can remove the factor $m^{1/2}$ in (1.4) unconditionally. In addition, our theorem shows that Selberg's bound (1.6) for $N(T,\chi)$ holds also for $N_1(T,\chi)$ as long as m is no greater than $\log qT/(\log\log qT)^3$.

2. LEMMAS

Throughout we assume $T>2$, and let $\rho'=\beta'+i\gamma'$ be a generic zero of $L'(s,\chi)$, where $s=\sigma+it$. Also, we set

$$G(s,\chi)=\frac{-m^s}{\chi(m)\log m}L'(s,\chi),$$

so that $G(s,\chi)$ is roughly 1 when $\Re s$ is sufficiently large, which can be seen by writing out its Dirichlet series expression. In fact, Yıldırım [13] showed that $G(s,\chi)$ is free of zeros and is dominated by 1 as long as $\sigma>1+\frac{m}{2}(1+\sqrt{1+\frac{4}{m\log m}})$. In practice it suffices to take $\sigma\geq 10m$, say, so that $|G(s,\chi)-1|<0.002$ uniformly in t and χ (cf. Lemma 3.1 of [11]).

Lemma 2.1. *Assume GRH. Let $T>2$ and suppose that $L(\sigma\pm iT,\chi)\neq 0$ and $L'(\sigma\pm iT,\chi)\neq 0$ for all $\sigma\in\mathbb{R}$. Then there exists an absolute large constant q_0 such that for $q>q_0$ we have*

$$N_1(T,\chi)=\frac{T}{\pi}\log\frac{qT}{2m\pi e}+A(T,\chi)+B(T,\chi)-A(-T,\chi)-B(-T,\chi)+O(1),$$

where

$$A(t,\chi)=\frac{1}{2\pi}\arg G(1/2+it,\chi),\qquad B(t,\chi)=\frac{1}{2\pi}\arg L(1/2+it,\chi),$$

and the argument is defined by continuous variation from $+\infty$, with the argument at $+\infty$ being 0.

Proof. Yıldırım proved that on GRH, if q is sufficiently large (larger than an absolute constant), then $L'(s,\chi)$ has at most one zero in the region $0<\sigma<1/2$ and, if it exists, the zeros is close to 0 (cf. Theorem 1 in [13]). Thus, if we let $N_1^*(T,\chi)$ denote the number of zeros of $L'(s,\chi)$ in $1/4<\sigma$, $|t|\le T$, then for q sufficiently large $N_1^*(T,\chi)=N_1(T,\chi)+O(1)$. Hence, we may assume q is large and estimate $N_1^*(T,\chi)$ instead.

We apply the Argument Principle to the function $G/L(s,\chi)$ along the rectangular contour with vertices $1/4-iT, 10m-iT, 10m+iT, 1/4+iT$, and obtain that the change in argument of G/L along this contour is equal to $2\pi(N_1^*(T,\chi)-N(T,\chi))$. Write Δ_L (respectively, Δ_R) for the change in argument of G/L along the part of the contour to the left (respectively, right) of the critical line. It is clear that

$$\begin{aligned}\Delta_R = &\arg G(1/2+iT,\chi)-\arg L(1/2+iT,\chi)\\ &-\arg G(1/2-iT,\chi)+\arg L(1/2-iT,\chi)+O(1).\end{aligned}$$

To estimate Δ_L, we start from the well-known formula (see (17) in Chapter 12 of [4])

$$\frac{L'}{L}(s,\chi)=-\frac{1}{2}\log\frac{q}{\pi}+B(\chi)-\frac{1}{2}\frac{\Gamma'}{\Gamma}(\frac{s+a}{2})+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right),$$

where $B(\chi)$ satisfies $\Re B(\chi)=-\Re\sum 1/\rho$ (cf. equation (18) in Chapter 12 of [4]), $a=0$ if $\chi(-1)=1$ and $a=1$ if $\chi(-1)=-1$, and the summation is over nontrivial zeros ρ of $L(s,\chi)$. Take real parts on both sides, we see that

$$\Re\frac{L'}{L}(s,\chi)=-\frac{1}{2}\log\frac{q}{\pi}-\frac{1}{2}\Re\frac{\Gamma'}{\Gamma}(\frac{s+a}{2})+\sum_\rho\frac{\sigma-1/2}{|s-\rho|^2}.$$

On the contour of Δ_L we have

$$\sum_\rho\frac{\sigma-1/2}{|s-\rho|^2}\le 0$$

since $\sigma\le 1/2$. Next, by Stirling's formula

$$-\frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right)=-\frac{1}{2}\log\left|\frac{s+a}{2}\right|+O\left(\frac{1}{|s+a|}\right)$$

for s on the contour of Δ_L. It follows that for such s

$$-\frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right)<C$$

for some absolute constant $C>0$. Collecting the above estimates, we obtain

$$\Re\frac{L'}{L}(s,\chi)<-\frac{1}{2}\log\frac{q}{\pi}+C,$$

and thus, there exists an absolute constant q_0 such that for $q > q_0$

$$\Re \frac{L'}{L}(s,\chi) < 0. \tag{2.1}$$

Therefore, for such q we arrive at

$$\begin{aligned}\Delta_L \arg \frac{G}{L} &= \Delta_L \arg \frac{L'}{L} + \Delta_L \arg \left(\frac{-m^s}{\chi(m)\log m} \right) \\ &= O(1) - 2T\log m.\end{aligned}$$

Combining the estimates for Δ_R and Δ_L we have

$$\begin{aligned}2\pi(N_1^*(T,\chi) - N(T,\chi)) = &\arg G(1/2+iT,\chi) - \arg L(1/2+iT,\chi) \\ &- \arg G(1/2-iT,\chi) + \arg L(1/2-iT,\chi) \\ &- 2T\log m + O(1).\end{aligned}$$

It follows from a standard argument that

$$\pi N(T,\chi) = T\log\frac{qT}{2\pi e} + \arg L(1/2+iT,\chi) - \arg L(1/2-iT,\chi) + O(1).$$

Inserting this to the above formula gives the lemma. □

Lemma 2.2. *Assume GRH. We have*

$$\arg L(\sigma \pm iT,\chi) \ll \frac{\log qT}{\log\log qT}$$

uniformly for $\sigma \geq 1/2$.

See Section 5 in [9], or exercise 11 of Section 13.2 in [8].

Lemma 2.3. *Assume GRH. We have*

$$\arg \frac{G}{L}(\sigma \pm iT,\chi) \ll \frac{m^{1/2}\log\log qT + m}{\sigma - 1/2}$$

for $1/2 < \sigma < 3$.

See Lemma 3.6 in [11].

Let us define

$$F_1(t,\chi) = \sum_{\beta'>1/2} \frac{\beta'-1/2}{(\beta'-1/2)^2 + (\gamma'-t)^2}.$$

We require the following result.

Lemma 2.4. *Assume GRH and order the ordinates of the zeros of* $L(s,\chi)$ *as* $\cdots < \gamma_{-1} < 0 < \gamma_1 \leq \gamma_2 \leq \cdots$. *Then*

$$\int_{\gamma_n}^{\gamma_{n+1}} F_1(t,\chi)dt \ll 1 + (\gamma_{n+1}-\gamma_n)\log q.$$

This is an immediate consequence of combining equation (3.5) in [11] and Lemma 3.5 in [11].

To state our next lemma we let $N_1(\mathscr{R})$ be the number of zeros of $L'(s,\chi)$ in the rectangular region $\mathscr{R}$ given by

$$T-H \le t \le T+H, \qquad 1/2 < \sigma \le 1/2 + H$$

where H is a parameter.

Lemma 2.5. *Assume GRH. We have*

$$N_1(\mathscr{R}) \ll N(T+H,\chi) - N(T-H,\chi) + 1 + H\log q.$$

Proof. Consider the integral

$$\mathcal{I} := \int_{T-H}^{T+H} F_1(t,\chi)dt.$$

From Lemma 2.4 we see that

$$\mathcal{I} \ll N(T+H,\chi) - N(T-H,\chi) + 1 + H\log q. \tag{2.2}$$

Thus, to prove the lemma, it suffices to show that

$$N_1(\mathscr{R}) \ll \mathcal{I}.$$

First observe that

$$F_1(t,\chi) \ge \sum_{\rho' \in \mathscr{R}} \frac{\beta' - 1/2}{(\beta'-1/2)^2 + (\gamma'-t)^2} \ ,$$

since each summand in the definition of $F_1(t,\chi)$ is positive. Hence,

$$\begin{aligned} \mathcal{I} &\ge \int_{T-H}^{T+H} \sum_{\rho' \in \mathscr{R}} \frac{\beta' - 1/2}{(\beta'-1/2)^2 + (\gamma'-t)^2} dt \\ &= \sum_{\rho' \in \mathscr{R}} \int_{T-H}^{T+H} \frac{\beta' - 1/2}{(\beta'-1/2)^2 + (\gamma'-t)^2} dt. \end{aligned}$$

Let $\theta(\rho') \in (0,\pi)$ be the argument of the angle at ρ' with two rays through $1/2 + i(T-H)$ and $1/2 + i(T+H)$ respectively. It is easy to see that

$$\theta(\rho') = \int_{T-H}^{T+H} \frac{\beta' - 1/2}{(\beta'-1/2)^2 + (\gamma'-t)^2} dt.$$

This gives us

$$\mathcal{I} \ge \sum_{\rho' \in \mathscr{R}} \theta(\rho').$$

Now for $\rho' \in \mathscr{R}$, we clearly have $\theta(\rho') \ge c$ for some absolute positive constants c. Therefore, we have

$$\mathcal{I} \ge cN_1(\mathscr{R}) \gg N_1(\mathscr{R}).$$

The result follows on combining this and (2.2). □

Lemma 2.6. *Let $s=\sigma+iT$. On GRH we have*

$$\frac{G'}{G}(s,\chi)=\sum_{|\rho'-s|<.5}\frac{1}{s-\rho'}+O(\log 2m\cdot\log qT),$$

uniformly for $1/2\le\sigma\le 2$.

Proof. Since $G'/G(10m+iT,\chi)\ll 1$, we may add and subtract it to get

$$G'/G(s,\chi)=G'/G(s,\chi)-G'/G(10m+iT,\chi)+O(1).$$

We then apply Hadamard factorization and obtain

$$G'/G(s,\chi)=\sum_{\rho'}\left(\frac{1}{s-\rho'}-\frac{1}{10m+iT-\rho'}\right)+O(1), \tag{2.3}$$

where the sum is over all zeros ρ' of $L'(s,\chi)$.

It is known that on GRH, for q larger than an absolute constant, the zeros of $L'(s,\chi)$ are among the following (cf. Theorem 1 in [13], Theorem 1.1 – 1.4 in [2].)

- Nontrivial zeros. On GRH, they lie on or to the right of the critical line $\Re s=1/2$.
- Trivial zeros. Except a possible zero to the right of the imaginary axis and close to 0, they all lie to the left of the imaginary axis, and form a rather regularly spaced discrete sequence clustering the negative real axis.

By standard methods it is easy to show that the contribution of trivial zeros in (2.3) is $O(\log m)$. For nontrivial zeros, we split the sum into several parts as follows. Write

$$\sum_{\rho'\text{nontrivial}}=\sum_{|\rho'-s|<.5}+\sum_{\rho'\in B_1}+\cdots+\sum_{\rho'\in B_N}+\sum_{\text{remaining }\rho'},$$

where $B_i=b_i-\cup_{j=0}^{i-1}b_j$ with b_0 being the disk $\{z:|z-s|<.5\}$ and b_i (for $i\ge 1$) the box

$$\{z:1/2\le\Re z\le 1/2+2^{i-1},\quad T-2^{i-1}\le\Im z\le T+2^{i-1}\},$$

and N is the smallest integer such that $2^{N-1} \geq 8m$. Write $M = 2^{N-1}$. Then we have

$$\begin{aligned}\sum_{\text{remaining } \rho'} \left(\frac{1}{s-\rho'} - \frac{1}{10m+iT-\rho'} \right) &\ll \sum_{\text{remaining } \rho'} \frac{m}{|s-\rho'| \cdot |10m+iT-\rho'|} \\ &= \sum_{n=0}^{\infty} \sum_{\substack{\beta' > 1/4 \\ 2^n M \leq |\gamma' - T| < 2^{n+1} M}} \frac{m}{|s-\rho'| \cdot |10m+iT-\rho'|} \\ &\ll \sum_{n=0}^{\infty} \left(\sum_{\substack{\beta' > 1/4 \\ 2^n M \leq |\gamma' - T| < 2^{n+1} M}} 1 \right) \cdot \frac{m}{(2^n M)^2}.\end{aligned}$$

For $q > q_0$, by (2.1) we see that if $L'(1/2+it, \chi) = 0$ then $L(1/2+it, \chi) = 0$. This shows that the number of ρ' lying on a segment on the critical line is no greater than the number of zeros of $L(s, \chi)$ on the same segment. Thus, we apply equation (1.2) and Lemma 2.5 and see that

$$\sum_{\substack{\beta' > 1/4 \\ 2^n M \leq |\gamma' - T| < 2^{n+1} M}} 1 \ll 2^n M \log q(T + 2^{n+1} M).$$

It follows that

$$\begin{aligned}\sum_{\text{remaining } \rho'} \left(\frac{1}{s-\rho'} - \frac{1}{10m+iT-\rho'} \right) &\ll \sum_{n=0}^{\infty} \frac{m}{2^n M} \log(2^n qTM) \\ &\ll \log qT.\end{aligned}$$

A similar argument shows that

$$\sum_{\rho' \in B_i} \left(\frac{1}{s-\rho'} - \frac{1}{10m+iT-\rho'} \right) \ll \log qT.$$

Thus, we have

$$\sum_{i=1}^{N} \sum_{\rho' \in B_i} \left(\frac{1}{s-\rho'} - \frac{1}{10m+iT-\rho'} \right) \ll \log(2m) \log(qT).$$

Finally, it is clear that

$$\sum_{|\rho' - s| < .5} \left(\frac{1}{s-\rho'} - \frac{1}{10m+iT-\rho'} \right) = \sum_{|\rho' - s| < .5} \frac{1}{s-\rho'} + O(\log qT).$$

The lemma now follows by collecting the above estimates. $\square$

3. PROOF OF THEOREM 1.1

Let q_0 be the absolute constant defined in Lemma 2.1. For $q \le q_0$ our theorem clearly follows from (1.7). Thus, we may assume $q > q_0$. Also, since $N_1(T,\chi)$ is right continuous with respect to T, it suffices to consider T's such that $L(\sigma \pm iT,\chi) \ne 0$ and $L'(\sigma \pm iT,\chi) \ne 0$ for any $\sigma \in \mathbb{R}$. It follows from Lemma 2.2 that GRH implies

$$\arg L(1/2 \pm iT,\chi) \ll \frac{\log qT}{\log\log qT}.$$

Thus, in view of Lemma 2.1, to prove the theorem it suffices to prove that

$$\arg G(1/2 \pm iT,\chi) \ll \frac{\log qT}{\log\log qT} + \sqrt{m\log m \log qT}. \tag{3.1}$$

We shall present the proof for $\arg G(1/2+iT,\chi)$, and $\arg G(1/2-iT,\chi)$ can be treated similarly.

Recall that $\arg G(1/2+iT,\chi)$ is defined by continuous variation along the horizontal line from $\infty + iT$ to $1/2+iT$ starting with the value 0. Let Δ_1 be the change in argument of G along the horizontal line from $\infty+iT$ to $1/2+X+iT$, and let Δ_2 be the change along the horizontal segment from $1/2+X+iT$ to $1/2+iT$, where $X \ll 1$ is a parameter to be determined later. We may require in addition that $X < 1/100$ for convenience.

To estimate Δ_1, we first apply Lemma 2.3 with $\sigma = 1/2+X$ to see that

$$\arg \frac{G}{L}(\sigma+iT,\chi) \ll \frac{m + m^{1/2}\log\log qT}{X}.$$

Then we use Lemma 2.2 and obtain that

$$\arg L(\sigma+iT,\chi) \ll \frac{\log qT}{\log\log qT}.$$

It follows that

$$\Delta_1 = \arg G(\sigma+iT,\chi) \ll \frac{\log qT}{\log\log qT} + \frac{m + m^{1/2}\log\log qT}{X}. \tag{3.2}$$

Next we bound Δ_2. First we observe that

$$\Delta_2 = \Im \int_{1/2}^{1/2+X} \frac{G'}{G}(\sigma+iT,\chi)\,d\sigma.$$

By Lemma 2.6 we have

$$\frac{G'}{G}(s,\chi) = \sum_{|\rho'-s|<.5} \frac{1}{s-\rho'} + O(\log 2m \log qT)$$

for s on the segment $[1/2+iT, 1/2+X+iT]$. It is convenient to modify this formula slightly. Let $\mathscr{D}$ be the disk centered at $1/2+X/2+iT$ with radius .5. We claim that

$$\frac{G'}{G}(s,\chi) = \sum_{\rho' \in \mathscr{D}} \frac{1}{s-\rho'} + O(\log 2m \log qT)$$

for s on the horizontal segment $[1/2+iT, 1/2+X+iT]$. Indeed, for such s and for ρ' not belonging to the intersection of $\mathscr{D}$ and the disk $|\rho'-s| < .5$, we see that $(s-\rho')^{-1} \ll 1$. Since the number of such ρ' is $O(\log qT)$, which can be seen from Lemma 2.5, their contribution is also $O(\log qT)$.

It now follows that

$$\Delta_2 = \Im \int_{1/2+iT}^{1/2+X+iT} \Bigg(\sum_{\rho' \in \mathscr{D}} \frac{1}{s-\rho'} + O(\log 2m \log qT) \Bigg) ds$$

(3.3)

$$\begin{aligned} &= \sum_{\rho' \in \mathscr{D}} \Bigg(\Im \int_{1/2+iT}^{1/2+X+iT} \frac{1}{s-\rho'}\, ds \Bigg) + O(X \log 2m \log qT) \\ &= \sum_{\rho' \in \mathscr{D}} \big[\arg(\tfrac{1}{2} + X + iT - \rho') - \arg(\tfrac{1}{2} + iT - \rho') \big] + O(X \log 2m \log qT) \\ &= \sum_{\rho' \in \mathscr{D}} f(\rho') \;+\; O(X \log 2m \log qT), \end{aligned}$$

say. Notice that $f(\rho')$ is plus or minus the argument of the angle subtended by the segments from $1/2+iT$ to ρ' and from $1/2+X+iT$ to ρ'. Thus, in particular, $f(\rho') \ll 1$.

We split the sum $\sum_{\rho' \in \mathscr{D}} f(\rho')$ into three parts. We let $\sum_1$ denote the sum over the $\rho' \in \mathscr{R}$, that is, the ρ' satisfying

$$T - H \le \gamma' \le T + H, \qquad 1/2 < \beta' \le 1/2 + H + iT,$$

where H is a parameter to be determined later. We let $\sum_2$ be the sum over the zeros $\rho' = 1/2 + i\gamma'$, if any, with

$$T - H \le \gamma' \le T + H.$$

Finally, we let $\sum_3$ denote the sum over the remaining ρ' in $\mathscr{D}$.

By Lemma 2.5 and our observation above that $f(\rho') \ll 1$, we see that

$$\begin{aligned} \sum\nolimits_1 &\ll \max |f(\rho')| \cdot N_1(\mathscr{R}) \\ &\ll N_1(\mathscr{R}) \\ &\ll N(T+H,\chi) - N(T-H,\chi) + 1 + H \log q. \end{aligned}$$

Recall that GRH implies that

$$N(T,\chi) = \frac{T}{\pi} \log \frac{qT}{2\pi e} + O\bigg(\frac{\log qT}{\log\log qT} \bigg).$$

Thus $N(T+H,\chi) - N(T-H,\chi) \ll \log qT/\log\log qT + H\log qT$. We therefore obtain

$$\sum\nolimits_1 \ll \log qT/\log\log qT + H\log qT. \tag{3.4}$$

For $q > q_0$, by (2.1) we see that if $L'(1/2+it,\chi) = 0$ then $L(1/2+it,\chi) = 0$. This shows that the number of ρ' involved in $\sum_2$ is no greater than the number of zeros of $L(s,\chi)$ on the same segment. Then by a similar argument for $\sum_1$, we see that

$$\sum\nolimits_2 \ll \log qT/\log\log qT + H\log qT. \tag{3.5}$$

It remains to estimate $\sum_3$. First we observe that

$$f(\rho') = \arg(1/2 + X + iT - \rho') - \arg(1/2 + iT - \rho') \ll \frac{X}{H}$$

for ρ' in $\sum_3$. Since the number of ρ' in $\mathscr{D}$ is at most $O(\log qT)$ by Lemma 2.5, we see that

$$\sum\nolimits_3 \ll \frac{X\log qT}{H}.$$

Combining this with (3.4) and (3.5), we see that

$$\sum_{\rho'\in\mathscr{D}} f(\rho') \ll \frac{\log qT}{\log\log qT} + H\log qT + \frac{X\log qT}{H}.$$

Thus, by (3.3) we obtain

$$\Delta_2 \ll \frac{\log qT}{\log\log qT} + H\log qT + \frac{X\log qT}{H} + X\log 2m\log qT.$$

Recall our estimate (3.2) that

$$\Delta_1 \ll \frac{\log qT}{\log\log qT} + \frac{m + m^{1/2}\log\log qT}{X}.$$

Hence, to prove (3.1) it suffices to show that

$$\Delta_1 + \Delta_2 \ll \frac{\log qT}{\log\log qT} + \sqrt{m\log m\log qT}$$

or,

$$\begin{aligned} &\frac{m + m^{1/2}\log\log qT}{X} + H\log qT + \frac{X\log qT}{H} + X\log 2m\log qT \\ &\ll \frac{\log qT}{\log\log qT} + \sqrt{m\log m\log qT}. \end{aligned} \tag{3.6}$$

If $m < (\log\log qT)^2$, we can take $X = (\log\log qT)^9(\log qT)^{-1}$ and $H = (\log\log qT)^6(\log qT)^{-1/2}$, say. If $(\log\log qT)^2 \le m \le \log qT(\log\log qT)^{-3}$, we take $X = .001\,m^{2/3}(\log qT)^{-2/3}$ and $H = 10\sqrt{X}$, say. Finally, if $m > \log qT(\log\log qT)^{-3}$, we take $X = .001\,m^{1/2}(\log qT\log m)^{-1/2}$ and $H = 10\sqrt{X}$. In all cases (3.6) can be easily verified. This completes our proof. □

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