NOTE ON THE NUMBER OF ZEROS OF $\zeta^{(k)}(s)$

FAN GE, ADE IRMA SURIAJAYA

ABSTRACT. Assuming the Riemann hypothesis, we prove that

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi e} + O_k\left(\frac{\log T}{\log\log T}\right),$$

where $N_k(T)$ is the number of zeros of $\zeta^{(k)}(s)$ in the region $0 < \Im s \le T$. We further apply our method and obtain a zero counting formula for the derivative of Selberg zeta functions, improving earlier work of Luo [10].

1. INTRODUCTION

Let $\zeta(s)$ be the Riemann zeta function, and let

$$N(T) := \sum_{\substack{0<\gamma\le T\\ \beta>0}} 1$$

be the zero counting function for $\zeta(s)$. Here and throughout, $\rho = \beta + i\gamma$ is a generic zero of $\zeta(s)$. It is known that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + E_0(T),$$

where

$$E_0(T) = \begin{cases} O(\log T), & \text{unconditionally,} \\ O\left(\frac{\log T}{\log\log T}\right), & \text{assuming the Riemann hypothesis (RH).} \end{cases} \tag{1}$$

The unconditional bound is known as the Riemann-von Mangoldt formula (see [17, Theorem 9.4]), and the conditional bound is due to J. E. Littlewood [9].

There has also been considerable interest in zeros of derivatives of $\zeta(s)$. Let $\zeta^{(k)}(s)$ be the k-th derivative of the Riemann zeta function, and let

$$N_k(T) := \sum_{\substack{0<\gamma_k\le T\\ \beta_k>0}} 1$$

be the zero counting function for $\zeta^{(k)}(s)$. Here and throughout, $\rho_k = \beta_k + i\gamma_k$ is a generic zero of $\zeta^{(k)}(s)$. In [2] B. C. Berndt proved that

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi e} + E_k(T)$$

where

$$E_k(T) = O_k(\log T).$$

This should be compared to the Riemann-von Mangoldt formula. In view of (1) one may expect to prove that, assuming RH,

$$E_k(T) = O_k\left(\frac{\log T}{\log\log T}\right) \tag{2}$$

for all positive integers k. The first result in this direction is due to H. Akatsuka [1], who showed that if RH is true then

$$E_1(T) = O\left(\frac{\log T}{\sqrt{\log\log T}}\right).$$

Yet this bound is weaker than (2). The second author [15] extended this estimate to higher derivatives and showed that on RH

$$E_k(T) = O_k\left(\frac{\log T}{\sqrt{\log\log T}}\right)$$

for all positive integers k.

Recently, the first author [5] was able to prove (2) for $k = 1$, namely,

$$E_1(T) = O\left(\frac{\log T}{\log\log T}\right).$$

A key ingredient in his proof is an upper bound for the number of zeros of $\zeta'(s)$ close to the critical line, and the idea there has its origin in Y. Zhang's work [18]. However, the method for $k = 1$ is not readily applicable for larger k. The purpose of this note is to modify the method in [5] and show that the estimate (2) holds for all positive integers k.

Theorem 1. *Assume RH. Then we have*

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi e} + O_k\left(\frac{\log T}{\log\log T}\right)$$

as $T \to \infty$.

We remark that Littlewood's conditional bound

$$E_0(T) = O\left(\frac{\log T}{\log\log T}\right) \tag{3}$$

was first proved in 1924. Later in 1944 A. Selberg [11] gave a different proof for this result. In 2007, D. A. Goldston and S. M. Gonek [7] showed that we can take the implied constant to be $1/2$. The current best known constant is $1/4$, and this is due to E. Carneiro, V. Chandee and M. B. Milinovich [3] who proved it using two different methods in 2013. It seems difficult to reduce the size of the bound (3), and this suggests that the bounds in Theorem 1 might be best possible within current knowledge.

On the other hand, using interesting heuristic arguments D. W. Farmer, S. M. Gonek and C. P. Hughes [4] have conjectured that $E_0(T) = O(\sqrt{\log T \log\log T})$. This raises the question of what bounds one should expect for $E_k(T)$. We have the following

Theorem 2. *Assume RH and suppose that $E_0(T) = O(\Phi(T))$ for some increasing function $\log\log T \ll \Phi(T) \ll \log T$. Then we have*

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi e} + O_k\big(\max\big\{\Phi(2T), \sqrt{\log T}\log\log T\big\}\big).$$

Clearly Theorem 1 is a consequence of Theorem 2, so we shall only prove the latter. We also remark that our method works well for some other zeta and L-functions in the T-aspect. In Section 4 we give a brief discussion on this. In particular, we prove an analogue of Weyl's law for the derivative of Selberg zeta functions.

2. Lemmas

Throughout, let $\Phi(T)$ be an increasing function satisfying $\log\log T \ll \Phi(T) \ll \log T$ and assume that $E_0(T) \ll \Phi(T)$. Further, we use the variables k and ℓ to denote orders of differentiation, where they are always positive integers.

We first express the error term of $N_k(T)$ in terms of arguments of certain functions.

Lemma 3. *Assume RH. Let $k \geq 2$ be an integer. For $T \geq 2$ satisfying $\zeta(\sigma + iT) \neq 0$ and $G_k(\sigma + iT) \neq 0$ for all $\sigma \in \mathbb{R}$, we have*

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{4\pi e} + \frac{1}{2\pi}\arg G_k(1/2 + iT) + \frac{1}{2\pi}\arg\zeta(1/2 + iT) + O_k(1),$$

where

$$G_k(s) = \frac{2^s(-1)^k}{(\log 2)^k}\zeta^{(k)}(s),$$

and the argument is defined by continuous variation from $+\infty$, with the argument at $+\infty$ being 0.

Proof. This is standard. Apply the argument principle to $\frac{G_k}{G_{k-1}}(s)$ on the rectangular region with vertices $1/4 + i, \sigma_k + i, \sigma_k + iT, 1/4 + iT$, where σ_k is large so that G_k is dominated by 1 to the right of σ_k. See also [15, Proposition 3.1] for an alternative proof. □

Lemma 4. *Assume RH and let $\ell \geq 1$ be an integer. Then for $1/2 + \frac{(\log\log T)^2}{\log T} < \sigma < 1$, we have*

$$\arg G_\ell(\sigma + iT) \ll_\ell \Phi(T) + \frac{\log\log T}{\sigma - \frac{1}{2}}.$$

Proof. This is follows from [15, Lemma 2.3] by taking $\epsilon_0 = (4\log T)^{-1}$ there. □

Lemma 5. *Let $\ell \geq 1$ be an integer. For all t sufficiently large we have*

$$\frac{G_\ell'}{G_\ell}(s) = \sum_{|\gamma_\ell - t| < 1}\frac{1}{s - \rho_\ell} + O_\ell(\log t),$$

uniformly for $1/2 \le \sigma \le 1$.

Proof. This can be proved in a standard way. See Theorem 9.6 (A) in [17] for example. $\square$

Lemma 6. *Assume RH and let* $\ell \ge 1$ *be an integer. Then*

$$\Re \frac{\zeta^{(\ell)}}{\zeta^{(\ell-1)}}(\sigma+it) < 0$$

holds for $0 < \sigma \le 1/2$ *and sufficiently large* t *whenever* $\zeta^{(\ell-1)}(\sigma+it) \ne 0$.

Proof. Put

$$\xi_\ell(s) := \Gamma\left(\frac{s}{2}\right)\zeta^{(\ell)}(s).$$

Then we have

$$\frac{\xi_{\ell-1}'}{\xi_{\ell-1}}(s) = \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta^{(\ell)}}{\zeta^{(\ell-1)}}(s).$$

Using Hadamard factorization we easily see that for large t,

$$\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{s+2n}\right) + O(1)$$

and

$$\frac{\zeta^{(\ell)}}{\zeta^{(\ell-1)}}(s) = \sum_{\rho_{\ell-1}}\left(\frac{1}{s-\rho_{\ell-1}} + \frac{1}{\rho_{\ell-1}}\right) + O(1),$$

where $\rho_{\ell-1}$ runs over all zeros of $\zeta^{(\ell-1)}(s)$. We can rewrite the latter as

$$\frac{\zeta^{(\ell)}}{\zeta^{(\ell-1)}}(s) = \left(\sum_{\beta_{\ell-1}\ge 1/2} + \sum_{\substack{\beta_{\ell-1}<1/2,\\ \gamma_{\ell-1}\ne 0}} + \sum_{\gamma_{\ell-1}=0}\right)\left(\frac{1}{s-\rho_{\ell-1}} + \frac{1}{\rho_{\ell-1}}\right) + O(1).$$

By [8, Corollary of Theorem 7], RH implies that $\zeta^{(\ell)}(s)$ has at most finitely many non-real zeros in $\Re(s) < 1/2$. This implies that the second sum is $O(1)$. Meanwhile [14] shows that

$$\sum_{\gamma_{\ell-1}=0}\left(\frac{1}{s-\rho_{\ell-1}} + \frac{1}{\rho_{\ell-1}}\right) = \sum_{j=1}^{\infty}\left(\frac{1}{s-(-2j+O(1))} + \frac{1}{-2j+O(1)}\right).$$

Thus

$$\begin{aligned}\frac{\xi_{\ell-1}'}{\xi_{\ell-1}}(s) &= \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta^{(\ell)}}{\zeta^{(\ell-1)}}(s)\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{2n} - \frac{1}{s+2n}\right) + O(1)\\ &\quad + \sum_{\beta_{\ell-1}\ge 1/2}\left(\frac{1}{s-\rho_{\ell-1}} + \frac{1}{\rho_{\ell-1}}\right) + \sum_{j=1}^{\infty}\left(\frac{1}{s-(-2j+O(1))} + \frac{1}{-2j+O(1)}\right) + O(1)\end{aligned}$$

$$= \sum_{\beta_{\ell-1}\geq 1/2}\left(\frac{1}{s-\rho_{\ell-1}}+\frac{1}{\rho_{\ell-1}}\right)+O(1)$$

when t is large.

Taking the real part, we have

$$\Re\frac{\xi_{\ell-1}'}{\xi_{\ell-1}}(s)=\sum_{\beta_{\ell-1}\geq 1/2}\frac{\sigma-\beta_{\ell-1}}{|s-\rho_{\ell-1}|^2}+O(1).$$

Hence using Stirling's formula for the Gamma function, we have

$$\Re\frac{\zeta^{(\ell)}}{\zeta^{(\ell-1)}}(\sigma+it)=\Re\frac{\xi_{\ell-1}'}{\xi_{\ell-1}}(\sigma+it)-\frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{\sigma+it}{2}\right)=\sum_{\beta_{\ell-1}\geq 1/2}\frac{\sigma-\beta_{\ell-1}}{|s-\rho_{\ell-1}|^2}-\frac{1}{2}\log t+O(1), \quad (4)$$

which is negative for $\sigma\leq 1/2$ and t large. □

Lemma 7. *Assume RH. Let $\mathcal{Z}_\ell=\{z_i\}_i$ be the collection of distinct ordinates of zeros of $\zeta,\zeta',...,\zeta^{(\ell)}$ on $\Re(s)=1/2$. For large T and $Y\leq T$, we have*

$$\sum_{T<z\leq T+Y,\, z\in\mathcal{Z}_\ell}1\ll\Phi(2T)+Y\log T.$$

Proof. It follows from Lemma 6 that for all $j\in\mathbb{N}$, zeros of $\zeta^{(j)}$ on the critical line at large heights can only occur at zeros of $\zeta^{(j-1)}$. Therefore, for sufficiently large T,

$$\sum_{T<z\leq T+Y,\, z\in\mathcal{Z}_\ell}1\leq\sum_{T<\gamma\leq T+Y}1\ll\Phi(2T)+Y\log T.$$

□

Write $\mathcal{D}=\mathcal{D}(T)$ for the region $\{w:\Re w\geq 1/2, |\Im w-T|\leq 1\}$. Divide $\mathcal{D}$ into N parts, as follows. Let $B_j=\{w:1/2\leq\Re w\leq 1/2+Y_j, |\Im w-T|\leq Y_j\}$ where $Y_j=2^jX$ and $X=(\log T)^{-1/2}$. We can write $\mathcal{D}=\cup_{j=1}^N R_j$ where $R_1=B_1$ and $R_j=(B_j-B_{j-1})\cap\mathcal{D}$ for $j\geq 2$. Note that $2^NX\approx 1$.

A key ingredient in [5] is an upper bound for the number of zeros of $\zeta'(s)$ in regions like R_j's. To prove our Theorem 2 we need such bounds for higher derivatives, and the following result provides us the desired estimates.

Lemma 8. *Let $N_k(R_j)$ be the number of zeros of $\zeta^{(k)}$ in R_j. Then $N_k(R_j)\ll_k Y_j\log T+\Phi(2T)$.*

Proof. Let R_j^* be R_j without the left side boundary on the critical line. In view of Lemma 7 it suffices to prove $N_k(R_j^*)\ll_k Y_j\log T+\Phi(2T)$. Denote by $\Theta(\rho_k;1/2+i(T+Y_j),1/2+i(T-Y_j))\in(0,\pi)$

the argument of the angle at ρ_k with two rays through $1/2+i(T-Y_j)$ and $1/2+i(T+Y_j)$. Note that $\Theta(\rho_k; 1/2+i(T+Y_j), 1/2+i(T-Y_j)) \gg 1$ if $\rho_k \in R_j^*$. Thus

$$\begin{aligned} N_k(R_j^*) &\ll \sum_{\rho_k \in R_j^*} \Theta(\rho_k; 1/2+i(T+Y_j), 1/2+i(T-Y_j)) \\ &= \sum_{\rho_k \in R_j^*} \int_{T-Y_j}^{T+Y_j} \frac{\beta_k - 1/2}{(\beta_k-1/2)^2+(\gamma_k-t)^2} dt \\ &= \int_{T-Y_j}^{T+Y_j} \sum_{\rho_k \in R_j^*} \frac{\beta_k - 1/2}{(\beta_k-1/2)^2+(\gamma_k-t)^2} dt \\ &\le \int_{T-Y_j}^{T+Y_j} \sum_{\beta_k > 1/2} \frac{\beta_k - 1/2}{(\beta_k-1/2)^2+(\gamma_k-t)^2} dt \\ &\le \sum_{\substack{T-Y_j \le z_i \le T+Y_j, \\ z_i \in \mathcal{Z}_k}} \int_{z_i}^{z_{i+1}} \sum_{\beta_k > 1/2} \frac{\beta_k - 1/2}{(\beta_k-1/2)^2+(\gamma_k-t)^2} dt. \end{aligned} \tag{5}$$

Write

$$F_k(t) = \sum_{\beta_k > 1/2} \frac{\beta_k - 1/2}{(\beta_k-1/2)^2+(\gamma_k-t)^2}.$$

Recall (4) that

$$F_k(t) = -\Re \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1/2+it) + O(\log t).$$

We claim that

$$\int_{z_i}^{z_{i+1}} F_k(t) dt \ll 1 + \log T \cdot (z_{i+1} - z_i).$$

To prove this, note that for t on the segment (z_i, z_{i+1}) we can write

$$\zeta^{(k)}(1/2+it) = \left(h\zeta \cdot \frac{\zeta'}{\zeta} \cdot \frac{\zeta''}{\zeta'} \cdots \frac{\zeta^{(k)}}{\zeta^{(k-1)}} \cdot \frac{1}{h} \right)(1/2+it)$$

where $h(s) = \pi^{-s/2}\Gamma(s/2)$. Thus, by using the temporary notation $\Delta \arg$ to denote the argument change along the segment (z_i, z_{i+1}), we have

$$\begin{aligned} \int_{z_i}^{z_{i+1}} F_k(t) dt &= \int_{z_i}^{z_{i+1}} \left(-\Re \frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1/2+it) + O(\log t) \right) dt \\ &= \left| \Delta \arg \zeta^{(k)}(1/2+it) \right| + O\left((z_{i+1}-z_i) \log t \right) \\ &= \left| \Delta \arg(h(1/2+it)\zeta(1/2+it)) + \Delta \arg \frac{\zeta'}{\zeta}(1/2+it) + \cdots + \Delta \arg \frac{\zeta^{(k)}}{\zeta^{(k-1)}}(1/2+it) \right. \\ &\qquad \left. + \Delta \arg \frac{1}{h(1/2+it)} \right| + ((z_{i+1}-z_i) \log t) \\ &\le \left| \Delta \arg(h(1/2+it)\zeta(1/2+it)) \right| + \sum_{l=1}^{k} \left| \Delta \arg \frac{\zeta^{(l)}}{\zeta^{(l-1)}}(1/2+it) \right| \\ &\qquad + \left| \Delta \arg \frac{1}{h(1/2+it)} \right| + ((z_{i+1}-z_i) \log t). \end{aligned}$$

It follows from the well-known functional equation for $h(s)\zeta(s)$ that

$$\Delta \arg(h(1/2+it)\zeta(1/2+it)) = 0.$$

From Lemma 6, we have

$$\Delta \arg \frac{\zeta^{(l)}}{\zeta^{(l-1)}}(1/2+it) \ll 1$$

for $l = 1, 2, \cdots, k$ and $t \in (z_i, z_{i+1})$. Moreover, by Stirling's formula we obtain

$$\begin{aligned}\left|\Delta \arg \frac{1}{h(1/2+it)}\right| &\ll \int_{z_j}^{z_{j+1}} \left|\frac{h'}{h}(1/2+it)\right| dt \\ &\ll (z_{j+1}-z_j)\log T.\end{aligned}$$

Thus

$$\int_{z_i}^{z_{i+1}} F_k(t)dt \ll_k 1 + (z_{i+1}-z_i)\log T,$$

as claimed.

It then follows from (5) and Lemma 7 that

$$\begin{aligned}N_k(R_j^*) &\ll_k \sum_{\substack{T-Y_j \le z_i \le T+Y_j,\\ z_i \in \mathcal{Z}_k}} (1 + (z_{i+1}-z_i)\log T) \\ &\ll \sum_{\substack{T-Y_j \le z_i \le T+Y_j,\\ z_i \in \mathcal{Z}_k}} 1 + (\log T) \cdot \sum_{\substack{T-Y_j \le z_i \le T+Y_j,\\ z_i \in \mathcal{Z}_k}} (z_{i+1}-z_i) \\ &\ll Y_j \log T + \Phi(2T).\end{aligned}$$

□

3. Proof of Theorem 2

Applying Lemma 3, we only need to show that

$$\arg G_k(1/2+iT) \ll_k \Phi(2T) + \sqrt{\log T}\log\log T$$

holds for all $k \in \mathbb{N}$.

Let $X = 1/\sqrt{\log T}$ as defined in the paragraph preceding Lemma 8. From Lemma 4, we see that

$$\arg G_k(1/2+X+iT) \ll_k \Phi(T) + \sqrt{\log T}\log\log T.$$

It remains to show

$$\Delta := \arg G_k(1/2+iT) - \arg G_k(1/2+X+iT) \ll \Phi(2T) + \sqrt{\log T}\log\log T. \tag{6}$$

From Lemma 5, we have

$$|\Delta| = \left|\Im \int_{1/2}^{1/2+X} \frac{G_k'}{G_k}(\sigma+iT)d\sigma\right|$$

$$\ll \sum_{|\gamma_k - T|<1} \Theta(\rho_k; 1/2+iT, 1/2+X+iT) + X\log T, \tag{7}$$

where $\Theta(a;b,c)$ is the (positive) angle at a in the triangle abc. Hence, it suffices to show that

$$\sum_{|\gamma_k - T|<1} \Theta(\rho_k; 1/2+iT, 1/2+X+iT) \ll \Phi(2T) + \sqrt{\log T}\log\log T.$$

From [8, Corollary of Theorem 7], RH implies that for sufficiently large T, $\zeta^{(k)}$ has no zeros in the left half of the critical strip above $T-1$. Hence we may assume that

$$\sum_{|\gamma_k - T|<1} \Theta(\rho_k; 1/2+iT, 1/2+X+iT) = \sum_{\rho_k\in\mathcal{D}} \Theta(\rho_k; 1/2+iT, 1/2+X+iT)$$

where $\mathcal{D}$ is the region defined in the paragraph preceding Lemma 8. Using the expression $\mathcal{D} = \cup_{j=1}^{N} R_j$ and Lemma 8 we have

$$\begin{aligned}
\sum_{\rho_k\in\mathcal{D}} \Theta(\rho_k; 1/2+iT, 1/2+X+iT) &= \sum_{j=1}^{N}\sum_{\rho_k\in R_j} \Theta(\rho_k; 1/2+iT, 1/2+X+iT) \\
&\ll \sum_{j=1}^{N} N_k(R_j)\frac{X}{Y_j} \\
&\ll_k \sum_{j=1}^{N} (2^j X\log T + \Phi(2T))\frac{1}{2^j} \\
&\ll \Phi(2T) + XN\log T.
\end{aligned}$$

Recall that $X = \frac{1}{\sqrt{\log T}}$ and $N \ll_k \log(1/X) \ll_k \log\log T$. Thus the above bound is

$$\ll_k \Phi(2T) + \sqrt{\log T}\log\log T$$

as desired. □

4. Other zeta and L-functions

Our method works well for some other zeta and L-functions in the T-aspect. Below we give two examples of the first derivative of Selberg zeta functions and Dirichlet L-functions, respectively. Dealing with higher derivatives of these functions would require information about the "trivial" zeros of those derivatives, which is not the purpose of this paper.

First, let us consider the Selberg zeta functions on cocompact hyperbolic surfaces. Precisely, let X be a compact Riemann surface of genus $g \geq 2$, and let $Z_X(s)$ be the associated Selberg zeta function. Denote by $\mathcal{N}(T)$ and $\mathcal{N}_1(T)$ the zero counting functions for $Z_X(s)$ and $Z_X'(s)$, respectively; so $\mathcal{N}(T)$ is the number of nontrivial zeros of $Z_X(s)$ up to height T, and similarly for $\mathcal{N}_1(T)$. Weyl's law tells us that

$$\mathcal{N}(T) = C_X T^2 + O\left(\frac{T}{\log T}\right)$$

where C_X is a specific constant depending on X. In [10] W. Luo proved that

$$\mathcal{N}_1(T) = C_X T^2 + O(T).$$

Following our method *in an identical manner*, we can prove that

$$\mathcal{N}_1(T) = C_X T^2 + D_X T + O\left(\frac{T}{\log T}\right), \tag{8}$$

where D_X is a specific constant depending on X. Thus (8) improves Luo's result. (Precisely, $C_X = g-1$ and $D_X = -\log N(P_{00})/2\pi$ where $N(P_{00}) = \min_{P_0} N(P_0)$; see page 1143 in [10] for more explanation of the notation.) The estimate (8) was proved by the first author (unpublished) using a different method, but our method here is simpler.

As another example, let $L(s,\chi)$ be the Dirichlet L-function where χ is a primitive Dirichlet character to the modulus q. Let $N(T,\chi)$ be the number of nontrivial zeros of $L(s,\chi)$ with heights between $-T$ and T. Define $N_1(T,\chi)$ similarly as the zero counting function for $L'(s,\chi)$. It follows from Selberg's work [12] that on the generalized Riemann hypothesis (GRH)

$$N(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2\pi e} + O\bigg(\frac{\log qT}{\log\log qT}\bigg). \tag{9}$$

As for $L'(s,\chi)$, recently the first author [6] proved that on GRH we have

$$N_1(T,\chi) = \frac{T}{\pi}\log\frac{qT}{2m\pi e} + O\bigg(\frac{\log qT}{\log\log qT} + \sqrt{m\log 2m\log qT}\bigg), \tag{10}$$

where m is the smallest prime number not dividing q. This improves earlier work of the second author [16]. Our method here should give analogues of (10) for higher derivatives of $L(s,\chi)$ once some standard information on trivial zeros of these derivatives is gathered. As a final remark, we note that in the T-aspect the error term in (10) is as good as that in (9). However, in the q-aspect $\sqrt{m\log 2m\log qT}$ might sometimes be larger than $\frac{\log qT}{\log\log qT}$. In fact, simple calculation shows that

$$\sqrt{m\log 2m\log qT} \ll \frac{\log qT}{\log\log qT}$$

when m is no greater than $\log qT/(\log\log qT)^3$, while the largest possible value for m is about $\log q$. It would be of interest to see if one can remove the second term in the error of (10).

Acknowledgement

This project was started when the first author was a postdoc fellow at the University of Waterloo and the second author was a member of iTHEMS under RIKEN Special Postdoctoral Researcher program. The second author is supported by JSPS KAKENHI Grant Number 18K13400.

References

[1] H. Akatsuka, *Conditional estimates for error terms related to the distribution of zeros of $\zeta'(s)$*, J. Number Theory **132** (2012), no. 10, 2242–2257.
[2] B. C. Berndt, *The number of zeros for $\zeta^{(k)}(s)$*, J. London Math. Soc. (2) **2** (1970), 577–580.
[3] E. Carneiro, V. Chandee and M. B. Milinovich, *Bounding $S(t)$ and $S_1(t)$ on the Riemann hypothesis*, Math. Ann. 356 (2013), 939-968.
[4] D. W. Farmer, S. M. Gonek and C. P. Hughes, *The maximum size of L-functions*, J. Reine Angew. Math. **609** (2007), 215–236.
[5] F. Ge, *The number of zeros of $\zeta'(s)$*, Int. Math. Res. Not. IMRN (2017), no. 5, 1578–1588.
[6] F. Ge, *The number of zeros of $L'(s,\chi)$*, Acta Arith. **190** (2019), no. 2, 127–138.

[7] D. A. Goldston and S. M. Gonek, *A note on $S(t)$ and the zeros of the Riemann zeta-function*, Bull. Lond. Math. Soc. 39 (2007), 482-486.
[8] N. Levinson and H. L. Montgomery, *Zeros of the derivative of the Riemann zeta-function*, Acta Math. **133** (1974) 49–65.
[9] J. E. Littlewood, *On the zeros of the Riemann zeta-function*, Proc. Camb. Philos. Soc. **22** (1924), 295–318.
[10] W. Luo, *On the zeros of the derivative of the Selberg zeta function*, Amer. J. Math. **127** (2005), no. 5, 1141–1151.
[11] A. Selberg, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$,* Avhandlinger utgitt av Det Norske Videnskaps-Akademi i Oslo I. Mat.-Naturv. Klasse (1944), No. 1, 1–27.
[12] A. Selberg, *Contributions to the theory of Dirichlet's L-functions*, Skr. Norske Vid.-Akad., Oslo I. (1946), no. 3, 1–62.
[13] A. Speiser, *Geometrisches zur Riemannschen Zetafunktion*, Math. Ann. **110** (1935), 514–521.
[14] R. Spira, *Another zero-free region for $\zeta^{(k)}(s)$*, Proc. Amer. Math. Soc. **26** (1970), 246–247.
[15] A. I. Suriajaya, *On the zeros of the k-th derivative of the Riemann zeta function under the Riemann hypothesis*, Funct. Approx. Comment. Math. **53** (2015), 69–95.
[16] A. I. Suriajaya, *Two estimates on the distribution of zeros of the first derivative of Dirichlet L-functions under the generalized Riemann hypothesis*, Journal de Théorie des Nombres de Bordeaux **29** (2017), no. 2, 471–502.
[17] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., edited by D. R. Heath-Brown; Oxford Science Publications, Oxford, 1986.
[18] Y. Zhang, *On the zeros of $\zeta'(s)$ near the critical line*, Duke Math. J. **110** (2001), 555-572.

Email address: fge@wm.edu

DEPARTMENT OF MATHEMATICS, COLLEGE OF WILLIAM & MARY, USA

Email address: adeirmasuriajaya@math.kyushu-u.ac.jp

FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, JAPAN