THE DISTRIBUTION OF ZEROS OF $\zeta'(s)$ AND GAPS BETWEEN ZEROS OF $\zeta(s)$

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ABSTRACT. Assume the Riemann Hypothesis. We establish a local structure theorem for zeros of the Riemann zeta-function $\zeta(s)$ and its derivative $\zeta'(s)$. As an application, we sharpen a result of Radziwiłł [18] concerning the global statistics of these zeros. Roughly speaking, we show that on the Riemann Hypothesis, if there occurs a small gap between consecutive zeta zeros, then there is exactly one zero of $\zeta'(s)$ lying not only very close to the critical line but also between that pair of zeta zeros. This refines a result of Zhang [23]. Some related results are also shown. For example, we prove a weak form of a conjecture of Soundararajan [19], and suggest a repulsion phenomena for zeros of $\zeta'(s)$.

1. INTRODUCTION

Throughout this paper $s = \sigma + it$ is a complex variable. Write $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ for a generic zero of $\zeta(s)$ and $\zeta'(s)$, respectively. If $\zeta(1/2 + i\gamma) = 0$, let γ^+ denote the smallest $t > \gamma$ with $\zeta(1/2 + it) = 0$. The phrase "γ is large" stands for "γ is larger than an absolute constant". Finally, we order the ordinates of zeros of $\zeta(s)$ as $0 < \gamma_1 \leq \gamma_2 \leq \cdots$, and similarly for zeros of $\zeta'(s)$.

The distribution of zeros of $\zeta'(s)$, and its relationship to zeros of $\zeta(s)$, has been investigated by many authors (see [1, 2, 5, 7, 8, 10, 13, 14, 15, 16, 18, 19, 20, 23]). For example, a well-known theorem of A. Speiser [20] states that the Riemann Hypothesis (RH) is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < 1/2$.

In [19] K. Soundararajan raised the following conjecture.

Conjecture. *(Soundararajan.) Assume RH. The following two statements are equivalent:*

$$\text{(A)} \quad \liminf_{\gamma' \to \infty} (\beta' - 1/2) \log \gamma' = 0;$$

$$\text{(B)} \quad \liminf_{\gamma \to \infty} (\gamma^+ - \gamma) \log \gamma = 0.$$

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As Soundararajan mentions, both of these assertions are probably true separately; the point of the conjecture is to indicate that the two assertions are related.

In his remarkable work, Y. Zhang [23] showed that on RH (B) implies (A), which solved one direction of Soundararajan's Conjecture. By contrast, the attempts to prove that (A) implies (B) have been unsuccessful. As an alternative, D. W. Farmer and H. Ki [5] considered the following. Let $w(x)$ be the indicator function of the unit interval $[0,1]$, and define

$$m'(v) = \liminf_{T\to\infty} \frac{2\pi}{T\log T} \sum_{T\le\gamma'\le 2T} w\left(\frac{(\beta'-1/2)\log T}{v}\right)$$

and

$$m(v) = \liminf_{T\to\infty} \frac{2\pi}{T\log T} \sum_{T\le\gamma_n\le 2T} w\left(\frac{(\gamma_{n+1}-\gamma_n)\log T}{2\pi v}\right).$$

They conjectured that if $m'(v) \gg v^{\alpha}$ for some $\alpha < 2$, then $m(v) > 0$ for all $v > 0$. This conjecture, which they viewed as a refinement of Soundararajan's conjecture, was recently proved (assuming RH) in an interesting paper by M. Radziwiłł [18].

On the other hand, by investigating random matrix models for $\zeta'(s)$, F. Mezzadri [17] conjectured an asymptotic formula for $m'(v)$. Namely,

$$m'(v) \sim \frac{8}{9\pi} v^{3/2} \quad \text{as } v \to 0. \tag{1}$$

We also refer the reader to [4] for a detailed study in this direction. The corresponding conjecture for $m(v)$ (see e.g. [5]) is

$$m(v) \sim \frac{\pi}{6} v^{3} \quad \text{as } v \to 0. \tag{2}$$

The validity of (1) and (2) would have significant consequences. In particular, from the work of J. B. Conrey and H. Iwaniec [3] and further work of Farmer and Ki [5] (see also Radziwiłł [18]), either of (1) or (2) (and even weaker estimates) would imply the non-existence of the Landau-Siegel zeros.

Notice that according to (1) and (2), $m'(v^2) \asymp m(v)$ as $v \to 0$, where the symbol $\asymp$ means $\ll$ and $\gg$. In fact, Radziwiłł [18] has conjectured that if at least one of $m(v)$ or $m'(v)$ is $\gg v^A$ for some $A > 0$, then $m(v/2\pi) \asymp m'(v^2)$, and he proved a slightly weaker form that $m(v^{1+\epsilon}) \ll m'(v^2) \ll m(v^{1-\epsilon})$. Our first result sharpens half of Radziwiłł's result.

Theorem 1. *Assume RH. There exists an absolute constant $c > 0$, such that*

$$m(v/2\pi) \le m'(v^2)$$

for all $0 < v < c$.

Note that in Theorem 1 we do not need the assumption of the lower bound for $m(v)$ or $m'(v)$. Theorem 1 is a consequence of the following result.

Theorem 2. *Assume RH. There exists an absolute constant $c > 0$ such that for any $v < c$ the following holds: for all large γ with $\gamma^+ - \gamma < v/\log\gamma$, the box*

$$\left\{ s = \sigma + it : \frac{1}{2} < \sigma < \frac{1}{2} + \frac{v^2}{4\log\gamma},\ \gamma \le t \le \gamma^+ \right\}$$

contains exactly one zero of $\zeta'(s)$. Moreover, the zero is not on the boundary of the box.

Theorem 2 can be viewed as a refinement of a result of Zhang (see Theorem 3 in [23]). Roughly speaking, Zhang's result states that on RH, if there exists a small gap between consecutive zeros of $\zeta(s)$, then there is a zero of $\zeta'(s)$ nearby. The main difference between Zhang's result and our Theorem 2 is that Theorem 2 provides a more accurate location of ρ': it places ρ' between consecutive zeros of $\zeta(s)$. Consequently, different pairs of consecutive ordinates γ, γ^+ correspond to different zeros ρ' of $\zeta'(s)$. This is the main ingredient of our proof of Theorem 1.

Theorem 2 should also be compared with a result of Soundararajan (see Lemma 11 in Section 2). Roughly speaking, Soundararajan's result states that there is *at most* one ρ' in a certain rectangular region between consecutive zeros of $\zeta(s)$. We will prove a complementary result (see Lemma 12 in Section 2), which asserts that there is *at least* one ρ' in a certain rectangular region. Apart from technical details, Theorem 2 is essentially a combination of these two results.

Our next result supports the assertion (A) implies (B) of Soundararajan's Conjecture.

Theorem 3. *Assume RH. For $\beta' > 1/2$ and $\gamma \le \gamma' < \gamma^+$, we have*

$$\gamma^+ - \gamma \ll \sqrt{(\beta' - 1/2)\log\gamma'}. \tag{3}$$

In particular, we have

$$\liminf_{\substack{\gamma'\to\infty \\ \beta'>1/2}} (\beta' - 1/2)(\log\gamma')^3 = 0 \Longrightarrow \liminf_{\gamma\to\infty} (\gamma^+ - \gamma)\log\gamma = 0. \tag{4}$$

This should be compared with a result of M. Z. Garaev and C. Y. Yıldırım [8], which states that on RH

$$\liminf_{\gamma'\to\infty} (\beta' - 1/2)(\log\gamma')(\log\log\gamma')^2 = 0 \Longrightarrow \liminf_{\gamma_n\to\infty} (\gamma_{n+1} - \gamma_n)\log\gamma_n = 0. \tag{5}$$

Note that the right-hand side of (5) would be trivially true if $\zeta(s)$ has infinitely many multiple zeros. This is not the case in (4) since γ^+ and γ are the ordinates of distinct zeros. However, we require a stronger assumption on the left-hand side of (4) than we do in (5).

It is also worth noting that equation (3) suggests a repulsion phenomena for zeros of $\zeta'(s)$. This comes from the repulsion between zeros of $\zeta(s)$ predicted by extrapolating from Montgomery's pair correlation conjecture, namely, $\gamma^+ - \gamma \gg \gamma^{-1/3+o(1)}$ (see e.g. [11]). Thus (3) suggests that

$$\beta' - 1/2 \gg (\gamma')^{-2/3+o(1)}$$

provided that $\beta' \neq 1/2$.

The paper is organized as follows. In Section 2 we state and prove a number of lemmas required for the proofs of our theorems. Then, in Section 3, we deduce Theorems 2, 1, and 3 in that order.

2. Preliminary results

Let $\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta'(s)$, where Γ is the usual Gamma-function. In [23] Zhang defined the following function $F(t)$, which played a key role in his proof:

$$F(t) = \begin{cases} -\Re \dfrac{\eta'}{\eta}\big(1/2+it\big), & \text{if } \eta(1/2+it) \neq 0, \\ \lim\limits_{v\to t} F(v), & \text{otherwise.} \end{cases} \tag{6}$$

The following proposition collects some results from [23].

Proposition 4. *(Zhang.)* (i) *The limit in (6) exists. Namely, the function $F(t)$ is well-defined. Moreover, $F(t)$ is continuous.*

(ii) *We have $F(t) = F_1(t) - F_2(t) + O(1)$, where*

$$F_1(t) = -\sum_{\beta'>1/2} \Re \frac{1}{1/2+it-\rho'}\ , \tag{7}$$

and

$$F_2(t) = \sum_{0<\beta'<1/2} \Re \frac{1}{1/2+it-\rho'}\ . \tag{8}$$

(iii) *If $\rho = 1/2 + i\gamma$ is a simple zero of $\zeta(s)$ with $\gamma > 0$, then*

$$F(\gamma) = \frac{1}{2}\log\gamma + O(1).$$

(iv) *We have*

$$\int_{\gamma}^{\gamma^+} F(t)dt \le \pi. \tag{9}$$

(v) *If both* $1/2 + i\gamma$ *and* $1/2 + i\gamma^+$ *are simple zeros of* $\zeta(s)$*, then*

$$\int_{\gamma}^{\gamma^+} F(t)dt \equiv 0 \pmod{\pi}.$$

For part (i) see equations (2.13) and (2.14) in [23]. For the remaining four parts, see Lemmas 2, 3 and 4 in [23].

For our purposes we require further properties of $F(t)$ given by the following Lemma 5, Lemma 7 and Lemma 8.

Lemma 5. *Let* $\rho = 1/2 + i\gamma$ *be a zero of* $\zeta(s)$ *with multiplicity* $m(\rho) = m$*. Then we have*

$$F(\gamma) = \frac{1}{2m}\log\gamma + O\big(\frac{1}{m}\big).$$

Proof. Following Zhang [23], we let $\xi(s) = h(s)\zeta(s)$ and $\eta(s) = h(s)\zeta'(s)$, where $h(s) = \pi^{-s/2}\Gamma(s/2)$. Note that by the functional equation, we have

$$i^n\xi^{(n)}(1/2 + it) \in \mathbb{R}$$

for any integer $n > 0$.

Suppose that $\rho = 1/2 + i\gamma$ is a zero of $\zeta(s)$ with multiplicity m. Then we have

$$\zeta(\rho) = \zeta'(\rho) = \cdots = \zeta^{(m-1)}(\rho) = 0,$$
$$\zeta^{(m)}(\rho) \neq 0.$$

It follows from Leibniz' law that

$$\begin{aligned} \xi(\rho) = \xi'(\rho) = \cdots = \xi^{(m-1)}(\rho) &= 0, \\ \eta(\rho) = \eta'(\rho) = \cdots = \eta^{(m-2)}(\rho) &= 0, \\ \text{and} \qquad \xi^{(m)}(\rho) = \eta^{(m-1)}(\rho) &\neq 0. \end{aligned}$$

In particular, we see that

$$i^m\eta^{(m-1)}(\rho) = i^m\xi^{(m)}(\rho) \in \mathbb{R}. \tag{10}$$

Here we work with real variables (to make the computation for the next lemma slightly clearer), as follows. We write

$$\eta(x+iy) = \eta(x,y)$$

with x and y real. If $\eta(s)$ is holomorphic at $s = x+iy$, then we have

$$\begin{aligned}\eta^{(k)}(x+iy) &= \frac{\partial^k}{\partial y^k}\eta(x,y)\cdot(i)^{-k} \\ &=: \eta_{y^k}(x,y)\cdot(i)^{-k}, \quad \forall k\in\mathbb{Z}^+. \end{aligned} \tag{11}$$

Since $\eta(\rho) = \eta'(\rho) = \cdots = \eta^{(m-2)}(\rho) = 0$ and $\eta^{(m-1)}(\rho) \neq 0$, it follows that

$$\eta(1/2,\gamma) = \eta_y(1/2,\gamma) = \cdots = \eta_{y^{m-2}}(1/2,\gamma) = 0,$$
$$\eta_{y^{m-1}}(1/2,\gamma) \neq 0.$$

Therefore, in a small neighborhood of γ, we can write

$$\eta(1/2,t) = (t-\gamma)^{m-1}\cdot p(t) \tag{12}$$

for some function p with $p(\gamma)\neq 0$. Hence, we have

$$\begin{aligned}\eta'(1/2+it) &= \eta_y(1/2,t)\cdot i^{-1} \\ &= \big((t-\gamma)^{m-1}p'(t) + (m-1)(t-\gamma)^{m-2}p(t)\big)\cdot i^{-1}.\end{aligned}$$

This gives

$$\frac{\eta'}{\eta}(1/2+it) = i^{-1}\frac{p'}{p}(t) + i^{-1}(m-1)(t-\gamma)^{-1},$$

and in particular,

$$-\Re\,\frac{\eta'}{\eta}(1/2+it) = -\Im\,\frac{p'}{p}(t).$$

Letting t tend to γ, we see that

$$F(\gamma) = -\Im\,\frac{p'}{p}(\gamma), \tag{13}$$

where the right-hand side makes sense since $p(\gamma)\neq 0$.

Now Leibniz' law gives us

$$\xi^{(m+1)}(\rho) = h\zeta^{(m+1)}(\rho) + (m+1)h'\zeta^{(m)}(\rho)$$

and

$$\xi^{(m)}(\rho) = h\zeta^{(m)}(\rho).$$

It follows that

$$\frac{\xi^{(m+1)}}{\xi^{(m)}}(\rho) = \frac{\zeta^{(m+1)}}{\zeta^{(m)}}(\rho) + (m+1)\frac{h'}{h}(\rho).$$

Similarly, we obtain

$$\frac{\eta^{(m)}}{\eta^{(m-1)}}(\rho) = \frac{\zeta^{(m+1)}}{\zeta^{(m)}}(\rho) + m\frac{h'}{h}(\rho).$$

Therefore, we have

$$\frac{\xi^{(m+1)}}{\xi^{(m)}}(\rho) - \frac{\eta^{(m)}}{\eta^{(m-1)}}(\rho) = \frac{h'}{h}(\rho). \tag{14}$$

On the other hand, by (11) and (12) we can easily compute that

$$\frac{\eta^{(m)}}{\eta^{(m-1)}}(\rho) = i^{-1} m \frac{p'}{p}(\gamma). \tag{15}$$

Combining (13), (14) and (15), taking the real part, and using Stirling's formula for the function h, we obtain

$$F(\gamma) = \frac{1}{m} \Re \frac{h'}{h}(\rho) = \frac{1}{2m} \log \gamma + O(1/m).$$

This completes the proof of Lemma 5. □

Lemma 6. *Let $\rho = 1/2 + i\gamma$ be a zero of $\zeta(s)$. Then we have*

$$\lim_{v \to \gamma} \arg \eta(1/2 + iv) \equiv \pi/2 \pmod{\pi}.$$

Here we only claim that the above limit exists modulo π.

Proof. The case that $\rho = 1/2 + i\gamma$ is a simple zero of $\zeta(s)$ has been treated by Zhang. (See the proof of Lemma 4 in [23].) Below we assume that $\rho = 1/2 + i\gamma$ is a multiple zero of $\zeta(s)$ with multiplicity $m > 1$. Thus in particular $\eta(1/2 + i\gamma) = 0$.

We use a temporary notation " $\lim$ " to denote either $\lim_{\substack{t_1 \to \gamma \\ t_1 > \gamma}}$ or $\lim_{\substack{t_1 \to \gamma \\ t_1 < \gamma}}$, but fixed throughout the proof.

By (12) we have

$$\Re\, \eta(1/2 + it_1) = \Re\, \eta(1/2, t_1) = (t_1 - \gamma)^{m-1} \cdot \Re\, p(t_1)$$

and

$$\Im\, \eta(1/2 + it_1) = \Im\, \eta(1/2, t_1) = (t_1 - \gamma)^{m-1} \cdot \Im\, p(t_1).$$

It follows that

$$\lim \frac{\Im\, \eta(1/2 + it_1)}{\Re\, \eta(1/2 + it_1)} = \lim \frac{\Im\, p(t_1)}{\Re\, p(t_1)}. \tag{16}$$

Recall that by (10) we have that

$$i^m \eta^{(m-1)}(\rho) \in \mathbb{R},$$

and by (11) that

$$\eta_{y^{m-1}}(1/2,\gamma)\cdot(i)^{-(m-1)}=\eta^{(m-1)}(\rho).$$

Thus, we obtain

$$i\cdot\eta_{y^{m-1}}(1/2,\gamma)\in\mathbb{R}.$$

Since $\eta_{y^{m-1}}(1/2,\gamma)=(m-1)!\,p(\gamma)$, we have $ip(\gamma)\in\mathbb{R}$, namely, $\Re\,p(\gamma)=0$. From this and the fact that $p(\gamma)\neq 0$, we see that $\Im\,p(\gamma)\neq 0$. Therefore, we have

$$\arctan\lim\frac{\Im\,p(t_1)}{\Re\,p(t_1)}\equiv\pi/2 \pmod{\pi}. \tag{17}$$

Then it is easy to see that

$$\begin{aligned}\lim\arg\eta(1/2+it_1)&\equiv\lim\arctan\frac{\Im\,\eta(1/2+it_1)}{\Re\,\eta(1/2+it_1)} \pmod{\pi}\\ &\equiv\arctan\lim\frac{\Im\,\eta(1/2+it_1)}{\Re\,\eta(1/2+it_1)} \pmod{\pi},\end{aligned}$$

and by (16) and (17), this is

$$\equiv\pi/2 \pmod{\pi}.$$

□

Lemma 7. *Assume RH. We have*

$$\int_{\gamma}^{\gamma^+}F(t)dt=\pi$$

for all large γ.

Proof. By part (i) of Proposition 4, we have

$$\int_{\gamma}^{\gamma^+}F(t)dt=\lim\int_{t_1}^{t_2}F(t)dt, \tag{18}$$

where the limit is taken for $t_1>\gamma, t_1\to\gamma$ and $t_2<\gamma^+, t_2\to\gamma^+$. Moreover, from the definition of $F(t)$ we easily have (see equation (2.16) in [23])

$$\int_{t_1}^{t_2}F(t)dt=\arg\eta(1/2+it_1)-\arg\eta(1/2+it_2). \tag{19}$$

It follows that

$$\int_{\gamma}^{\gamma^+}F(t)dt=\lim\arg\eta(1/2+it_1)-\lim\arg\eta(1/2+it_2).$$

Thus by Lemma 6 we have

$$\int_{\gamma}^{\gamma^+}F(t)dt\equiv 0 \pmod{\pi}. \tag{20}$$

By part (ii) of Proposition 4,

$$F(t) = F_1(t) - F_2(t) + O(1).$$

This comes from the Hadamard factorization for $\eta(s)$. With a little more care we can actually get

$$F(t) = F_1(t) - F_2(t) - C + O(1/t)$$

for some constant C. If we assume RH, then by Speiser's theorem $\zeta'(s)$ has no zeros in $0 < \sigma < 1/2$. Thus by definition (8) we have $F_2(t) = 0$. Therefore, assuming RH, we have

$$F(t) = F_1(t) - C + O(1/t). \tag{21}$$

The expression for C is easily computed to be

$$C = \Re \sum_{\beta'>0} \frac{1}{\rho'} + \sum_{n=1}^{\infty} \left(\frac{1}{\rho'_n} + \frac{1}{2n} \right) - \frac{\log \pi}{2} - \frac{C_0}{2} - 2 + \frac{\zeta''}{\zeta'}(0), \tag{22}$$

where C_0 is Euler's constant, and $\rho'_n \in (-2n-2, -2n)$ is a real zero of $\zeta'(s)$ (see Theorem 9 in [16]).

We can show that $C < 0$ (unconditionally). In fact, by equation (4) in [22], we have

$$\Re \sum_{\beta'>0} \frac{1}{\rho'} < 0.185.$$

Also, using the numerical values $\rho'_1 > -2.72$, $\rho'_2 > -5$ and $\rho'_3 > -7.1$ (see e.g. Table 1 in [6]), and using the trivial bound $\rho'_n > -2n-2$ for the remaining ρ'_n, we get the following inequality, which is crude but sufficient for our purpose

$$\sum_{n=1}^{\infty} \left(\frac{1}{2n} + \frac{1}{\rho'_n} \right) < 0.4.$$

Finally, inserting the numerical values for other constants in (22), namely,

$$-\frac{\log \pi}{2} < -0.57, \qquad -\frac{C_0}{2} < -0.28,$$
$$\frac{\zeta''}{\zeta'}(0) < 2.19, \quad \text{(see equation (3.6) in [12])}$$

we get $C < 0$.

By (21) we see that for all large t

$$F(t) > F_1(t) \geq 0. \tag{23}$$

It follows that for large γ

$$\int_{\gamma}^{\gamma^+} F(t)dt > 0.$$

Combining this with (9) and (20), we obtain

$$\int_{\gamma}^{\gamma^{+}} F(t)dt = \pi$$

for all large γ. $\square$

For $\Re(\rho') > 1/2$, let $\theta(\rho', t_1, t_2) \in (0, \pi)$ be the argument of the angle at ρ' with rays through $1/2+it_1$ and $1/2+it_2$ respectively. This notation will be used in the proof of Lemma 8, Lemma 9 and Theorem 3. We also recall the following result of B. Berndt [1]. Let $N_1(T)$ be the number of zeros of $\zeta'(s)$ in the region $0 < \Im s \leq T$. Then we have

$$N_1(T) = \frac{T}{2\pi} \log \frac{T}{4\pi e} + O(\log T). \tag{24}$$

We also remark that on RH the error term in (24) can be reduced to $O\big(\frac{\log T}{\log\log T}\big)$ (see [9]).

The following lemma shows that one can evaluate the constant C in equation (21). We remark that in this paper it does not matter what the exact value of C is; nevertheless, we present our argument here since it might be of interest for future research.

Lemma 8. *Assume RH, and let $F_1(t)$ be defined in* (7)*. Then we have*

$$F(t) = F_1(t) + \log 2/2 + O(1/t).$$

Proof. Suppose T is large, and consider the integral

$$\int_T^{2T} F(t)dt. \tag{25}$$

Let N_d be the number of distinct zeros of $\zeta(s)$ with $T \leq \gamma \leq 2T$. Order the ordinates of distinct zeros of $\zeta(s)$ in $[T, 2T]$ as $T \leq g_1 < g_2 < \cdots < g_{N_d} \leq 2T$. Apply Lemma 7 and (23), we can write the above integral as

$$\int_T^{g_1} + \int_{g_1}^{g_2} + \cdots + \int_{g_{N_d-1}}^{g_{N_d}} + \int_{g_{N_d}}^{2T} F(t)dt = \sum_{j=1}^{N_d} \int_{g_j}^{g_{j+1}} F(t)dt + O(1). \tag{26}$$

Next, we estimate the integral (25) in a different way. By (21) we have

$$\int_T^{2T} F(t)dt = \int_T^{2T} F_1(t)dt - CT + O(1).$$

By the definition of $F_1(t)$,

$$\int_T^{2T} F_1(t)dt = -\Re \sum_{\beta'>1/2} \int_T^{2T} \frac{1}{1/2+it-\rho'}dt = \sum_{\beta'>1/2} \theta(\rho', T, 2T).$$

Hence, we obtain

$$\int_T^{2T} F(t)dt = \sum_{1/2<\beta'} \theta(\rho', T, 2T) - CT + O(1). \tag{27}$$

We claim that

$$\sum_{1/2<\beta'} \theta(\rho', T, 2T) = \pi N_d - T\log 2/2 + O(T/\log T). \tag{28}$$

To see this, we separate the sum over ρ' into five parts. We let

$$\begin{aligned}
S_1 &= \{\rho' : \beta' > 1/2,\, T + \log^2 T \le \gamma' \le 2T - \log^2 T\},\\
S_2 &= \{\rho' : \beta' > 1/2,\, 2T - \log^2 T \le \gamma' \le 2T + \log^2 T\},\\
S_3 &= \{\rho' : \beta' > 1/2,\, 2T + \log^2 T \le \gamma' \le 3T\},\\
S_4 &= \{\rho' : \beta' > 1/2,\, 3T \le \gamma'\},\\
S_5 &= \{\rho' : \beta' > 1/2,\, \gamma' \le T + \log^2 T\}.
\end{aligned}$$

For ρ' in S_1, it is clear that

$$\theta(\rho', T, 2T) = \pi + O(1/\log^2 T).$$

Therefore, we have

$$\sum_{\rho' \in S_1} \theta(\rho', T, 2T) = \pi|S_1| + O(|S_1|\log^{-2} T),$$

and by (24) this is

$$\sum_{\rho' \in S_1} \theta(\rho', T, 2T) = \pi|S_1| + O(T/\log T).$$

Similarly, we can easily show that

$$\begin{aligned}
&\sum_{\rho' \in S_2} \theta(\rho', T, 2T) \ll |S_2| \ll \log^3 T,\\
&\sum_{\rho' \in S_3} \theta(\rho', T, 2T) \ll |S_3|\log^{-2} T \ll \frac{T}{\log T}.
\end{aligned}$$

To estimate the sum over S_4, we note that if $\gamma' \in [nT, (n+1)T]$ for some $n \geq 3$, then $\theta(\rho', T, 2T) \ll 1/Tn^2$. It follows that

$$\begin{aligned}\sum_{\rho' \in S_4} \theta(\rho', T, 2T) &= \sum_{n=3}^{\infty} \sum_{\substack{\beta' > 1/2 \\ nT \leq \gamma' < (n+1)T}} \theta(\rho', T, 2T) \\ &\ll \sum_{n=3}^{\infty} \sum_{\substack{\beta' > 1/2 \\ nT \leq \gamma' < (n+1)T}} \frac{1}{Tn^2} \\ &\ll \sum_{n=3}^{\infty} T \log(nT) \cdot \frac{1}{Tn^2} \\ &\ll \log T.\end{aligned}$$

Finally, we can separate S_5 into three parts in the same manner as $S_2 \cup S_3 \cup S_4$, and the above discussion clearly implies that

$$\sum_{\rho' \in S_5} \theta(\rho', T, 2T) \ll \log^3 T + \frac{T}{\log T} + \log T \ll \frac{T}{\log T}.$$

Collecting the above estimates, we arrive at

$$\begin{aligned}\sum_{1/2 < \beta'} \theta(\rho', T, 2T) &= \pi |S_1| + O(T/\log T) \\ &= \pi \cdot \big(\#\{\rho' : \beta' > 1/2,\ T \leq \gamma' \leq 2T\} + O(\log^3 T)\big) \\ &\qquad + O(T/\log T) \\ &= \pi \cdot \#\{\rho' : \beta' > 1/2,\ T \leq \gamma' \leq 2T\} + O(T/\log T).\end{aligned}$$

Let N_1 denote the number of zeros of $\zeta'(s)$ lying in the horizontal strip $[T, 2T]$, N_{10} be the number of zeros of $\zeta'(s)$ lying on the segment $[T, 2T]$ on the critical line, and N be the number of zeros of $\zeta(s)$ on the same segment. Using (24) and familiar estimates for the zero counting function of $\zeta(s)$, we have

$$N_1 - N = -\frac{\log 2}{2\pi} T + O(\log T).$$

Thus, we see that

$$\begin{aligned}\#\{\rho' : \beta' > 1/2,\ T \leq \gamma' \leq 2T\} &= N_1 - N_{10} \\ &= (N - N_{10}) + (N_1 - N) \\ &= N_d - \frac{\log 2}{2\pi} T + O(T/\log T).\end{aligned}$$

Hence the claim (28) follows.

Combining (27) and (28) we get

$$\int_T^{2T} F(t)dt = \pi N_d - T\log 2/2 - CT + O(T/\log T).$$

This together with (26) gives

$$\sum_{j=1}^{N_d}\Big(\pi - \int_{g_j}^{g_{j+1}} F(t)dt\Big) = \Big(C + \frac{\log 2}{2}\Big)T + O(T/\log T).$$

By Lemma 7, the left-hand side is 0 for all large T. Thus we have $C = -\log 2/2$. The lemma now follows from (21). □

Lemma 9. *Assume RH. We have*

$$\sum_{1/2<\beta'} \theta(\rho', \gamma, \gamma^+) + \frac{\log 2}{2}(\gamma^+ - \gamma) + O\Big(\frac{\gamma^+ - \gamma}{\gamma}\Big) = \pi$$

for all large γ.

Proof. By the definition of $F_1(t)$ we have

$$\int_\gamma^{\gamma^+} F_1(t)dt = -\Re \sum_{\beta'>1/2} \int_\gamma^{\gamma^+} \frac{1}{1/2 + it - \rho'}dt = \sum_{\beta'>1/2} \theta(\rho', \gamma, \gamma^+).$$

Using this, the result then follows immediately from Lemma 7 and Lemma 8. □

Next we state two useful results of Soundararajan.

Lemma 10. *For* $\beta' > 1/2$ *and* $\gamma' > 0$*, we have*

$$\big|\rho - \rho'\big| \geq \sqrt{2(\beta' - 1/2)/\log\gamma}.$$

See Lemma 2.1 in [19].

Lemma 11. *Assume RH. The box*

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 1/\log\gamma,\ \gamma \leq t \leq \gamma^+\}$$

contains at most one zero of $\zeta'(s)$.

See Proposition 1.6 in [19].

The following result is also required, which can be viewed as a partial complement to Lemma 11.

Lemma 12. *Assume RH. Let a be any constant less than $\pi/3$. There exists a constant $\gamma_0(a)$ such that for $\gamma > \gamma_0(a)$ with $\gamma^+ - \gamma < a/\log\gamma$, the box*

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 2.5(\gamma^+ - \gamma), \gamma < t < \gamma^+\}$$

contains a zero of $\zeta'(s)$.

Proof. Let γ be large. Let $\mathscr{O} = 1/2 + i(\gamma + \gamma^+)/2$. That is, $\mathscr{O}$ is the middle point of ρ and ρ^+. Write $\gamma^+ - \gamma = 2l$ and $d = 5l$. Also write

$$F_1(t) = F_{11}(t) + F_{12}(t), \tag{29}$$

where

$$F_{11}(t) = -\sum_{\substack{\beta'>1/2 \\ |\rho'-\mathscr{O}|<d}} \Re\frac{1}{1/2 + it - \rho'} = \sum_{\substack{\beta'>1/2 \\ |\rho'-\mathscr{O}|<d}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2}\ ,$$

and

$$F_{12}(t) = -\sum_{\substack{\beta'>1/2 \\ |\rho'-\mathscr{O}|\geq d}} \Re\frac{1}{1/2 + it - \rho'} = \sum_{\substack{\beta'>1/2 \\ |\rho'-\mathscr{O}|\geq d}} \frac{\beta' - 1/2}{(\beta' - 1/2)^2 + (\gamma' - t)^2}\ .$$

In the rest of the proof for the lemma we assume $\gamma \leq t \leq \gamma^+$.

For $|\rho' - \mathscr{O}| \geq d$, we have

$$|\rho' - (1/2 + it)| \geq |\rho' - \rho| - |\rho - (1/2 + it)| \geq |\rho' - \rho|/2$$

and

$$|\rho' - (1/2 + it)| \leq |\rho' - \rho| + |\rho - (1/2 + it)| \leq 3|\rho' - \rho|/2.$$

It follows that

$$4F_{12}(\gamma)/9 \leq F_{12}(t) \leq 4F_{12}(\gamma).$$

In particular, the second inequality gives

$$F_{12}(t) \leq 2\log\gamma + O(1) \tag{30}$$

in view of Lemma 5.

Now suppose that there is no zero of $\zeta'(s)$ in the box

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 2.5(\gamma^+ - \gamma), \gamma < t < \gamma^+\}. \tag{31}$$

Then we may write

$$F_{11}(t) = f(t) + g(t), \tag{32}$$

where

$$f(t) = \sum_{\substack{\beta'>1/2 \\ |\rho'-\mathscr{O}|<d \\ \gamma'\geq\gamma^+}} \frac{\beta'-1/2}{(\beta'-1/2)^2+(\gamma'-t)^2} ,$$

and

$$g(t) = \sum_{\substack{\beta'>1/2 \\ |\rho'-\mathscr{O}|<d \\ \gamma'\leq\gamma}} \frac{\beta'-1/2}{(\beta'-1/2)^2+(\gamma'-t)^2} .$$

Observe that for $\gamma \leq t \leq \gamma^+$ we have $f(t) \leq f(\gamma^+)$ and $g(t) \leq g(\gamma)$. It follows that

$$F_{11}(t) \leq f(\gamma^+) + g(\gamma) \leq F_1(\gamma^+) + F_1(\gamma).$$

Applying Lemma 5, we obtain

$$F_{11}(t) \leq \log\gamma + O(1). \tag{33}$$

Combining (29), (33) and (30), we get

$$F_1(t) \leq 3\log\gamma + O(1)$$

for $t \in [\gamma, \gamma^+]$. It follows that

$$\int_\gamma^{\gamma^+} F_1(t)dt \leq 3(\gamma^+ - \gamma)\log\gamma + O(\gamma^+ - \gamma).$$

On the other hand, by Lemma 7 we have

$$\int_\gamma^{\gamma^+} F(t)dt = \pi,$$

and by Lemma 8 this is

$$\int_\gamma^{\gamma^+} F_1(t)dt = \pi + O(\gamma^+ - \gamma).$$

Hence, we obtain

$$\pi \leq 3(\gamma^+ - \gamma)\log\gamma + O(\gamma^+ - \gamma).$$

But since we are assuming $\gamma^+ - \gamma < a/\log\gamma$ (with $a < \pi/3$ being a constant), the above inequality would give

$$\pi \leq 3a + O(\gamma^+ - \gamma).$$

This can not hold if γ is large enough (depending on a). Hence the assumption (31) is false for large γ. This completes our proof. □

Combining Lemma 11 and Lemma 12 we immediately obtain

Corollary 13. *Assume RH. For large γ with $\gamma^+ - \gamma < 0.4/\log\gamma$, the box*

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 1/\log\gamma, \gamma \le t \le \gamma^+\}$$

contains exactly one zero of $\zeta'(s)$. Moreover, the zero is not on the boundary of the box.

Remark. In the proof of Lemma 12, if we skip the step of decomposing $F_1(t)$ into $F_{11}(t)$ and $F_{12}(t)$ as in (29), but decompose $F_1(t)$ just in the manner of (32), then we can prove the following result, whose proof we shall omit.

Proposition 14. *Assume RH. Let Δ be any constant less than π. There exists a constant $\gamma_0(\Delta)$ such that for $\gamma > \gamma_0(\Delta)$ with $\gamma^+ - \gamma < \Delta/\log\gamma$, the strip $\{s = \sigma + it : \gamma < t < \gamma^+\}$ contains a zero of $\zeta'(s)$.*

3. Proofs of Theorems 2, 1 and 3

Proof of Theorem 2. Let $v < 0.4$ and assume $\gamma^+ - \gamma < v/\log\gamma$. By Corollary 13, the box

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + 1/\log\gamma, \gamma \le t \le \gamma^+\}$$

contains exactly one zero of $\zeta'(s)$, say $\rho' = \beta' + i\gamma'$, and it is not on the boundary. Thus, to prove the theorem, it suffices to prove that

$$\beta' - 1/2 < \frac{v^2}{4\log\gamma}\ . \tag{34}$$

Without loss of generality, we may assume $\gamma^+ - \gamma' \ge \gamma' - \gamma$, and hence $\gamma^+ - \gamma \ge 2(\gamma' - \gamma)$. We shall also keep in mind that $\beta' - 1/2 < 1/\log\gamma$.

By Lemma 10, we have

$$|\rho' - \rho| \ge \sqrt{\frac{2(\beta' - 1/2)}{\log\gamma}}.$$

This gives

$$(\gamma' - \gamma)^2 \ge \frac{2(\beta' - 1/2)}{\log\gamma} - (\beta' - 1/2)^2 \ge \frac{\beta' - 1/2}{\log\gamma}$$

where the second inequality follows from the fact that $\beta' - 1/2 < 1/\log\gamma$. Hence, we get

$$\frac{v^2}{\log^2\gamma} > (\gamma^+ - \gamma)^2 \ge 4(\gamma' - \gamma)^2 \ge \frac{4(\beta' - 1/2)}{\log\gamma}.$$

This gives (34) and completes the proof. $\square$

Proof of Theorem 1. The proof is a simple counting argument. We supply the details for completeness. Take c to be the same as in Theorem 2, and let $v < c/2$. Define

$$\mathscr{S} = \big\{n : T \le \gamma_n \le 2T, \gamma_{n+1} - \gamma_n \le \frac{v}{\log T}\big\}$$

and

$$\mathscr{T} = \big\{m : T \le \gamma'_m \le 2T, \beta'_m - \frac{1}{2} \le \frac{v^2}{\log T}\big\}.$$

Recall that

$$m'(v) = \liminf_{T\to\infty} \frac{2\pi}{T\log T} \sum_{T\le\gamma'\le 2T} w\bigg(\frac{(\beta'-1/2)\log T}{v}\bigg)$$

and

$$m(v) = \liminf_{T\to\infty} \frac{2\pi}{T\log T} \sum_{T\le\gamma_n\le 2T} w\bigg(\frac{(\gamma_{n+1}-\gamma_n)\log T}{2\pi v}\bigg),$$

where $w(x)$ is the indicator function of $[0,1]$. Thus, to prove $m(v/2\pi) \le m'(v^2)$, it suffices to show that $|\mathscr{S}| - 1 \le |\mathscr{T}|$ for all large T.

Write $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2$, where

$$\mathscr{S}_1 = \{n : T \le \gamma_n \le 2T, \gamma_{n+1} - \gamma_n = 0\},$$

and

$$\mathscr{S}_2 = \big\{n : T \le \gamma_n \le 2T, 0 < \gamma_{n+1} - \gamma_n \le \frac{v}{\log T}\big\}.$$

Similarly, write $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$, where

$$\mathscr{T}_1 = \{k : T \le \gamma'_k \le 2T, \beta'_k - \frac{1}{2} = 0\},$$

and

$$\mathscr{T}_2 = \big\{k : T \le \gamma'_k \le 2T, 0 < \beta'_k - \frac{1}{2} \le \frac{v^2}{\log T}\big\}.$$

We clearly have $|\mathscr{S}| = |\mathscr{S}_1| + |\mathscr{S}_2|$ and $|\mathscr{T}| = |\mathscr{T}_1| + |\mathscr{T}_2|$. Moreover, it is easy to see that there is a bijection between $\mathscr{S}_1$ and $\mathscr{T}_1$. Namely, we have $|\mathscr{S}_1| = |\mathscr{T}_1|$.

Let α be the largest element in $\mathscr{S}_2$. We show that there is an injective mapping from $\mathscr{S}_2 - \{\alpha\}$ to $\mathscr{T}_2$. This will give $|\mathscr{S}_2| - 1 \le |\mathscr{T}_2|$, and therefore completes the proof.

By the definition of $\mathscr{S}_2$, we have $\gamma_{n_1} > \gamma_{n_2}$ if $n_1, n_2 \in \mathscr{S}_2$ and $n_1 > n_2$. Thus, if $n \in \mathscr{S}_2 - \{\alpha\}$, then $\gamma_n < \gamma_\alpha$, which implies that $\gamma_n^+ \le \gamma_\alpha \le 2T$.

Take any $n \in \mathscr{S}_2 - \{\alpha\}$. It follows from the definition of $\mathscr{S}_2$ that

$$\gamma_n^+ - \gamma_n \le v/\log T \le 2v/\log\gamma_n.$$

Applying Theorem 2, we see that the box

$$\{s = \sigma + it : 1/2 < \sigma < 1/2 + v^2/\log \gamma_n, \gamma_n < t < \gamma_n^+\}$$

contains a zero of $\zeta'(s)$, say $\rho'_{k(n)}$. Since

$$0 < \beta'_{k(n)} - \frac{1}{2} < \frac{v^2}{\log \gamma_n} \le \frac{v^2}{\log T}, \quad \text{and} \quad T \le \gamma_n < \gamma'_{k(n)} < \gamma_n^+ \le 2T,$$

it follows that $k(n) \in \mathscr{T}_2$.

We take the mapping $\phi : \mathscr{S}_2 - \{\alpha\} \longrightarrow \mathscr{T}_2$ via $\phi(n) = k(n)$. It remains to show that ϕ is injective. Suppose this is not the case, then we would have $k(n_1) = k(n_2)$ for some $n_1 > n_2$. But that would imply

$$\gamma'_{k(n_1)} > \gamma_{n_1} \ge \gamma_{n_2}^+ > \gamma'_{k(n_2)} = \gamma'_{k(n_1)},$$

a contradiction. This completes the proof. $\square$

Proof of Theorem 3. Let $\rho' = \beta' + i\gamma'$ be a zero of $\zeta'(s)$ with $\beta' > 1/2$ and γ' large. Let γ and γ^+ be such that $\gamma \le \gamma' < \gamma^+$.

By Lemma 9 we have

$$\sum_{1/2<\Re\lambda'} \theta(\lambda', \gamma, \gamma^+) + \frac{\log 2}{2}(\gamma^+ - \gamma) + O\Big(\frac{\gamma^+ - \gamma}{\gamma}\Big) = \pi,$$

where the sum is over all zeros λ' of $\zeta'(s)$ with real part greater than 1/2. In particular, this gives

$$\theta(\rho', \gamma, \gamma^+) < \pi - \frac{\log 2}{2}(\gamma^+ - \gamma) + O\Big(\frac{\gamma^+ - \gamma}{\gamma}\Big).$$

Thus, it follows that

$$\theta_1 + \theta_2 > \frac{\log 2}{2}(\gamma^+ - \gamma) + O\Big(\frac{\gamma^+ - \gamma}{\gamma}\Big) \gg \gamma^+ - \gamma,$$

where θ_1 and θ_2 are the angles at ρ and ρ^+, respectively, of the triangle (ρ, ρ', ρ^+).

By Lemma 10

$$|\rho - \rho'| \gg \sqrt{(\beta' - 1/2)/\log \gamma}.$$

Therefore, we see that

$$\sin \theta_1 = \frac{\beta' - 1/2}{|\rho - \rho'|} \ll \sqrt{(\beta' - 1/2) \log \gamma},$$

and this gives

$$\theta_1 \ll \sqrt{(\beta' - 1/2) \log \gamma}.$$

Similarly, we have

$$\theta_2 \ll \sqrt{(\beta' - 1/2) \log \gamma}.$$

Thus, we see that

$$\gamma^+ - \gamma \ll \theta_1 + \theta_2 \ll \sqrt{(\beta' - 1/2) \log \gamma}.$$

This completes our proof. □

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